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GEOMETRICAL SIGNIFICANCE OF ISOTHERMAL CONJUGACY OF A NET OF CURVES.

By E. J. Wilczynski.

Introduction.

Let

$$Ddu^2 + 2D'dudv + D''dv^2 \tag{1}$$

be the second fundamental differential form of a surface S, and let us consider a region R on this surface which is free from parabolic points so that, for all points in R,

$$D'^2 - DD'' \neq 0. \tag{2}$$

If D' is equal to zero for all points of R, the curves $u =$ const. and $v =$ const. form a *conjugate* net. If this condition is satisfied, and if besides the ratio $D : D''$ assumes the form of a function of u alone multiplied by a function of v alone, so that

$$\frac{\partial^2 \log D/D''}{\partial u \partial v} = 0, \qquad D' = 0, \tag{3}$$

the net is said to be *isothermally* conjugate. This name is due to Bianchi,* and was chosen by him because, in all such cases, it is possible to choose new variables

$$\bar{u} = \varphi(u), \qquad \bar{v} = \psi(v)$$

in such a way as to transform (1) into the isothermal form

$$\lambda(\bar{u}, \bar{v})(d\bar{u}^2 + d\bar{v}^2),$$

without changing the conjugate net under consideration.

Bianchi also proved that the property of isothermal conjugacy is of a *projective* character.† That is, if an isothermally conjugate net is subjected to any projective transformation, the resulting net will again be isothermally conjugate. But Bianchi did not furnish any geometric interpretation of the analytic conditions (3) which serve to define such systems. Moreover, although the importance of this notion was becoming more and more apparent, because of a steadily increasing body of theorems which made use of it, no serious attempt seems to have been made to discover its true significance until 1915, when the author of the present paper discovered an algebraic relation, between certain completely interpreted projective in-

* L. Bianchi, "Lezioni di geometria differenziale" (Seconda edizione), Vol. 1, p. 168.

† Ibid. p. 169.

variants, which is characteristic of isothermally conjugate systems.* Thus, in a sense, the problem was solved. But the solution was not altogether satisfying because it lacked simplicity and could not be formulated completely in terms of purely descriptive relations. A year afterward, the late G. M. Green, whose premature death has deprived geometry of one of its most brilliant students, took a long step in advance.† In fact, Green believed that he had settled the matter completely. But he had overlooked an important case in which his geometric criterion fails to distinguish between isothermally conjugate nets and nets of an entirely different kind.

The present paper was written for the purpose of completing the solution of this problem, as nearly as possible in the spirit of Green's method, and making use of Green's notations. I dedicate this paper to his memory.

1. RÉSUMÉ AND REVISION OF GREEN'S THEORY.

Let

$$y^{(k)} = y^{(k)}(u, v), \qquad (k = 1, 2, 3, 4) \tag{4}$$

be the homogeneous coördinates of a point P_y. When the variables, u and v, vary over their ranges, P_y will in general describe a surface S_y. We shall assume that this surface does not degenerate into a curve, and that it is non-developable. If the curves $u =$ const. and $v =$ const. form a conjugate net on S_y, there exists a completely integrable system of differential equations of the form

$$\begin{aligned} y_{uu} &= ay_{vv} + by_u + cy_v + dy, \qquad a \neq 0, \\ y_{uv} &= \ast + b'y_u + c'y_v + d'y, \end{aligned} \tag{5}$$

whose fundamental, linearly independent, solutions are $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$. Conversely, every completely integrable system of form (5) defines a non-developable surface referred to a conjugate net.

The integrability conditions of system (5) teach us that there exists a function p, of u and v, such that‡

$$p_u = b + 2c', \qquad p_v = \frac{2ab' - c - a_v}{a}. \tag{6}$$

Consequently we can make a transformation of the form

$$y = \lambda \bar{y}, \tag{7}$$

* E. J. Wilczynski, "The General Theory of Congruences," *Transactions of the American Mathematical Society*, Vol. 16 (1915), p. 323. Quoted hereafter as W.

† G. M. Green, "Projective Differential Geometry of One-parameter Families of Space Curves and Conjugate Nets on a Curved Surface (Second Memoir), AM. JOUR. OF MATH., Vol. 38 (1916), p. 323. Quoted hereafter as Green (Second Memoir).

‡ G. M. Green, "Projective Differential Geometry of One-parameter Families of Space Curves, etc. (First Memoir), AM. JOUR. OF MATH., Vol. 37 (1915), p. 223. Quoted hereafter as Green (First Memoir).

where λ is subjected to the conditions

$$\frac{\lambda_u}{\lambda} = \tfrac{1}{4}p_u, \qquad \frac{\lambda_v}{\lambda} = \tfrac{1}{4}p_v. \tag{8}$$

The resulting system of differential equations has the same form as (5), with the coefficients*

$$\begin{aligned} &A = a, \qquad B = b - \tfrac{1}{2}p_u, \qquad C = c + \frac{a}{2}p_v, \\ &D = d + \tfrac{1}{4}bp_u + \tfrac{1}{4}cp_v - \tfrac{1}{4}p_{uu} + \tfrac{1}{4}ap_{vv} - \tfrac{1}{16}p_u^2 + \tfrac{1}{16}ap_v^2, \\ &B' = b' - \tfrac{1}{4}p_v, \qquad C' = c' - \tfrac{1}{4}p_u, \\ &D' = d' + \tfrac{1}{4}b'p_u + \tfrac{1}{4}c'p_v - \tfrac{1}{4}p_{uv} - \tfrac{1}{16}p_up_v. \end{aligned} \tag{9}$$

These coefficients are *seminvariants* of (5), and the new system is said to be in its *canonical form*. The relations

$$B + 2C' = 0, \qquad 2AB' - C - A_v = 0, \tag{10}$$

which follow from (9), are characteristic of this canonical form.

Any proper transformation of the form

$$\bar{u} = \varphi(u), \qquad \bar{v} = \psi(v), \tag{11}$$

affects only the parametric representation of the conjugate net given by (5), but leaves the net itself unchanged. The *invariants* of the net are those functions of the seminvariants, which remain unchanged by transformations of form (11), except for a factor. The fundamental invariants are†

$$\begin{aligned} &\mathfrak{A} = A, \qquad \mathfrak{B}' = B' - \frac{3}{8}\frac{A_v}{A}, \qquad \mathfrak{C}' = C' + \frac{1}{8}\frac{A_u}{A}, \\ &\mathfrak{D}' = D' + B'C', \qquad \mathfrak{D} = D - (B'A_v - AB'_v) - C'_u + 3(AB'^2 - C'^2); \end{aligned} \tag{12}$$

besides these, the following two, the Laplace-Darboux invariants of the net,‡

$$H = D' + B'C' - B_u, \qquad K = D' + B'C' - C'_v \tag{13}$$

are especially important.

The curves $u =$ const. and $v =$ const. of our conjugate net are not asymptotic lines. Therefore, the osculating planes of the two curves of the net, which meet at a point P_y of the surface, determine, as their line of intersection, a line passing through P_y and not in the tangent plane. This line is called the *axis* of P_y, and the totality of all such lines is called the *axis congruence* of the given conjugate system. The developables of the

* Green (First Memoir), p. 224.

† Ibid., p. 226.

‡ Ibid., p. 231–232.

axis congruence correspond to a net of curves on S_y, called the *axis curves*, whose differential equation is*

$$(14)\quad a\left(K + 2b'_u - b_v - \frac{\partial^2 \log a}{\partial u \partial v}\right) du^2 - \mathfrak{D} du dv - (H + 2b'_u - b_v) dv^2 = 0,$$

where a, b, and b' may be replaced by A, B, B' and where the relations (10) may then be used. The *anti-axis curves* are defined by*

$$(15)\quad a\left(K + 2b'_u - b_v - \frac{\partial^2 \log a}{\partial u \partial v}\right) du^2 + \mathfrak{D} du dv - (H + 2b'_u - b_v) dv^2 = 0.$$

Their tangents at any point of the surface are the harmonic conjugates of the axis curve tangents with respect to the tangents of the original conjugate system $u =$ const., $v =$ const.

The covariants

$$(16)\qquad \rho = y_u - c'y, \qquad \sigma = y_v - b'y$$

are the variables which determine the Laplace transformations of system (5). The points P_ρ and P_σ are in the plane tangent to S_y at P_y. The locus of P_ρ is the second sheet of the focal surface of the congruence formed by the tangents of the curves $v =$ const. on S_y. P_σ is connected in the same way with the congruence of tangents of the curves $u =$ const. on S_y. The line $P\rho P\sigma$, which moreover corresponds to the axis of P_y by duality, is called the *ray of* P_y. The totality of rays, for all surface points, is called the *ray congruence*, and the curves on S_y which correspond to the developables of the ray congruence, are called its *ray curves*.* The differential equation of the ray curves is

$$(17)\qquad aH du^2 - \mathfrak{D} du dv - K dv^2 = 0.$$

The *anti-ray curves* are related to the ray curves in the same way as the axis curves to the anti-axis curves. Their differential equation is as follows†;

$$(18)\qquad aH du^2 + \mathfrak{D} du dv - K dv^2 = 0.$$

There exists a uniquely determined conjugate net on the surface such that the two tangents of this new net, at any point of the surface, shall separate not only the asymptotic tangents, but also the tangents of the original conjugate system, harmonically. Green has called this system of curves the *associate conjugate net*,‡ and found its differential equation to be

$$(19)\qquad a du^2 - dv^2 = 0,$$

the asymptotic net of S_y being determined by

$$(20)\qquad a du^2 + dv^2 = 0.$$

* W., pp. 314–316 and Green (Second Memoir), pp. 308 and 310.

† W., pp. 317–318 and Green (Second Memoir), p. 309.

‡ Green (Second Memoir), p. 313.

In the case of an isothermally conjugate net, a has the form of a product of a function of u alone by a function of v alone, so that

$$\frac{\partial^2 \log a}{\partial u \partial v} = 0, \qquad a \neq 0. \tag{21}$$

It will then be possible to find a transformation

$$\bar{u} = U(u), \qquad \bar{v} = V(v),$$

such that the value of a in the transformed differential equations becomes equal to unity. Thus, *if the parametric net is isothermally conjugate, we may assume*

$$a = 1. \tag{22}$$

Let us consider the three quadratics (14), (18), and (19). The Jacobian of (14) and (19) is

$$a\mathfrak{D}du^2 + 2a\left(H - K + \frac{\partial^2 \log a}{\partial u \partial v}\right)dudv + \mathfrak{D}dv^2 = 0; \tag{23}$$

the Jacobian of (18) and (19) is

$$a\mathfrak{D}du^2 + 2a(H - K)dudv + \mathfrak{D}dv^2 = 0, \tag{24}$$

and clearly these Jacobians are equivalent, as quadratics in $du : dv$, if (21) is satisfied. But they are also equivalent if $\mathfrak{D} = 0$, and this is the case which Green failed to consider. In this exceptional case the axis curves and ray curves are so related to the parametric conjugate system that at every surface point the tangents belonging to the latter are separated harmonically by the tangents of each of the former nets, unless still other invariants vanishing cause one or both of these nets to become indeterminate. On account of these properties, let us call such conjugate nets, characterized by the condition $\mathfrak{D} = 0$, *harmonic conjugate nets.*

We have proved the following theorem.

THEOREM 1. *A conjugate net whose axis tangents, anti-ray tangents, and associate conjugate tangents, form three pairs of an involution at every point of the net, is either isothermally conjugate, or harmonic, or both.*

In this theorem, the axis tangents and anti-ray tangents may be replaced simultaneously by the anti-axis tangents and ray tangents, respectively.

Since it is our purpose to characterize isothermally conjugate nets completely by geometric properties, we must now search for properties of such nets which they do *not* share with harmonic conjugate nets. In most cases the following theorem will enable us to distinguish between harmonic and isothermally conjugate nets.

THEOREM 2. *The involution, mentioned in Theorem 1, has the parametric*

conjugate tangents as its double lines, if and only if the original net is harmonic. Therefore, the given net is isothermally conjugate, and not harmonic, if the three pairs of tangents mentioned in Theorem 1 are pairs of an involution, and if, besides, the double lines of this involution do not coincide with the parametric tangents.

If, however, the double elements of this involution *do* coincide with the parametric tangents, we can only conclude that the given net is harmonic. It may or may not be isothermally conjugate, at the same time. Thus our geometric criterion fails to distinguish between nets which are both isothermally conjugate and harmonic, and those which are merely harmonic.

Green* has shown that the associate conjugate net of an isothermally conjugate net is also isothermally conjugate, and vice versa, a theorem which we shall generalize in the next section. We may, therefore, apply theorems 1 and 2 to the associate conjugate net, obtaining the following result.

THEOREM 3. *If a conjugate net is isothermally conjugate, the associate conjugate net is also isothermally conjugate and vice versa. Consequently, the associate axis tangents, the associate anti-ray tangents, and the conjugate tangents of the original net, at any point of the net, will form three pairs of an involution. The double lines of this second involution will coincide with the associate conjugate tangents if and only if the associate conjugate net is harmonic.*

The associate axis tangents, etc., mentioned in this theorem, are related to the associate conjugate system in the same manner as the axis tangents, etc. are to the original system. By combining theorems 1, 2, 3, we obtain the following criterion.

THEOREM 4. *For an isothermally conjugate net both of the involutions, mentioned in theorems 1 and 3 exist. Conversely, if both of these involutions exist for a conjugate net, we can conclude that the net is isothermally conjugate unless both the original net and its associate net are harmonic.*

2. PENCILS OF CONJUGATE NETS ON A SURFACE.

Theorem 4 seems to be the most comprehensive criterion which can be obtained without introducing something essentially new into the discussion, but it does not solve the problem completely. For, it does not enable us to distinguish geometrically between conjugate nets which are harmonic, possess a harmonic associate net, and are besides isothermally conjugate, and conjugate nets which possess merely the first two of these properties. In order to solve our problem completely we introduce a new notion, that of a *pencil of conjugate systems*, a notion which we shall introduce at present only in connection with our special problem but which seems to be one of considerable general importance.

* Green (Second Memoir), p. 324.

Let us assume that the given conjugate system is isothermally conjugate, let the independent variables be chosen so that $a = 1$, and let the equations (5) be taken in their canonical form. Then we shall have

$$
(25)\qquad
\begin{aligned}
y_{uu} &= y_{vv} + By_u + Cy_v + Dy, \qquad a = A = 1,\\
y_{uv} &= \ \ * \ \ + B'y_u + C'y_v + D'y,
\end{aligned}
$$

where, on account of (10),

$$
(26)\qquad B = -\,2C', \qquad C = 2B'.
$$

The differential equations of the original conjugate system will be $dudv = 0$, that of the associate system will be $du^2 - dv^2 = 0$, and that of the asymptotic lines will be $du^2 + dv^2 = 0$. The differential equation

$$
(27)\qquad \alpha du^2 + 2\beta dudv + \gamma dv^2 = 0
$$

will determine a conjugate net if and only if

$$
\alpha + \gamma = 0,
$$

a condition obtained by equating to zero the harmonic invariant of (27) and $du^2 + dv^2 = 0$, the differential equation of the asymptotic lines. The tangents, at any point, of the curves of such a conjugate net will divide the corresponding tangents of the original conjugate net in a *constant* cross-ratio, if and only if the ratio of α to β is a constant. Consequently, *the differential equation*

$$
(28)\qquad (du + kdv)(kdu + dv) = 0,
$$

where k is an arbitrary constant, will determine a one-parameter family of conjugate nets each of which has the property that, at every point, the two tangents which belong to it determine a constant cross-ratio with those which belong to the original net.

We shall speak of the one-parameter family of conjugate nets, determined in this way by a given one, as a *pencil of conjugate nets.* There is one such net for every value, real or complex, of the constant k, but it is clear from (28) that the same net will correspond to two values of k which are negative reciprocals of each other. The net which corresponds to $k = 0$ or $k = \infty$ is the original net, and that which corresponds to the values $k = \pm 1$ is the associate net. For $k = \pm i$ the two factors of (28) become identical with each other and with one of the factors of $du^2 + dv^2$; the net degenerates into one of the families of asymptotic lines counted twice and therefore is not, properly speaking, a net at all. By a *proper* net of the pencil we mean any one of its nets excepting the two just mentioned which correspond to $k = i$ and $k = -\,i$. Of course every proper net of a pencil may be regarded as determining, in its turn, a pencil of nets.

But it follows at once, from the definition of a pencil, that all of these pencils coincide with each other and with the original pencil, and further that the nets of a pencil may be arranged in pairs associate to each other.

In order to study the properties of an individual net of the pencil, we introduce the variabels

$$\bar{u} = u - kv, \qquad \bar{v} = ku + v \tag{29}$$

into (25) in place of u and v. These variables will be independent if $1 + k^2$ is different from zero. We shall assume

$$1 + k^2 \neq 0, \tag{30}$$

a hypothesis which excludes from consideration only the improper conjugate systems formed by each of the two sets of asymptotic lines. We find

$$\begin{aligned} y_u &= y_{\bar{u}} + ky_{\bar{v}}, \qquad y_v = -ky_{\bar{u}} + y_{\bar{v}}, \\ y_{uu} &= y_{\bar{u}\bar{u}} + 2ky_{\bar{u}\bar{v}} + k^2 y_{\bar{v}\bar{v}}, \\ y_{uv} &= -ky_{\bar{u}\bar{u}} + (1 - k^2)y_{\bar{u}\bar{v}} + ky_{\bar{v}\bar{v}}, \\ y_{vv} &= k^2 y_{\bar{u}\bar{u}} - 2ky_{\bar{u}\bar{v}} + y_{\bar{v}\bar{v}}. \end{aligned} \tag{31}$$

Substitution of these values into equations (25) gives

$$\begin{aligned} (1 - k^2)y_{\bar{u}\bar{u}} + 4ky_{\bar{u}\bar{v}} - (1 - k^2)y_{\bar{v}\bar{v}} &= B\,(y_{\bar{u}} + ky_{\bar{v}}) + C\,(-ky_{\bar{u}} + y_{\bar{v}}) + Dy, \\ -ky_{\bar{u}\bar{u}} + (1 - k^2)y_{\bar{u}\bar{v}} + ky_{\bar{v}\bar{v}} &= B'(y_{\bar{u}} + ky_{\bar{v}}) + C'(-ky_{\bar{u}} + y_{\bar{v}}) + D'y, \end{aligned}$$

whence

$$\begin{aligned} (1 + k^2)^2(y_{\bar{u}\bar{u}} - y_{\bar{v}\bar{v}}) &= (1 - k^2)[(B - kC)y_{\bar{u}} + (kB + C)y_{\bar{v}} + Dy] \\ &\quad - 4k[(B' - kC')y_{\bar{u}} + (kB' + C')y_{\bar{v}} + D'y], \\ (1 + k^2)^2 y_{\bar{u}\bar{v}} &= k[(B - kC)y_{\bar{u}} + (kB + C)y_{\bar{v}} + Dy] \\ &\quad + (1 - k^2)[(B' - kC')y_{\bar{u}} + (kB' + C')y_{\bar{v}} + D'y]. \end{aligned} \tag{32}$$

These equations show, in the first place, that the new conjugate net, $\bar{u} =$ const., $\bar{v} =$ const., is isothermally conjugate, giving

THEOREM 5. *An isothermally conjugate net determines a pencil, all of whose proper nets are isothermally conjugate.*

This theorem includes, as a special case, Green's theorem that the associate net of an isothermally conjugate net is also isothermally conjugate. But we may draw a still farther reaching conclusion, by remembering that the same pencil of nets is determined if we start from any one of its proper nets in place of the one actually used. We then obtain the following result.

THEOREM 6. *If a pencil of conjugate nets contains one isothermally conjugate net, then all proper nets of the pencil are isothermally conjugate.*

We may reduce (32) to the form (25) by dividing by $(1 + k^2)^2$. If we denote the corresponding coefficients by A_k, B_k, C_k, etc., we find

$$
(33)\qquad
\begin{aligned}
(1+k^2)^2B_k &= (1-k^2)(B-kC) - 4k(B'-kC'),\\
(1+k^2)^2C_k &= (1-k^2)(kB+C) - 4k(kB'+C'),\\
(1+k^2)^2D_k &= (1-k^2)D - 4kD',\\
(1+k^2)^2B'_k &= k(B-kC) + (1-k^2)(B'-kC'),\\
(1+k^2)^2C'_k &= k(kB+C) + (1-k^2)(kB'+C'),\\
(1+k^2)^2D'_k &= kD + (1-k^2)D',
\end{aligned}
$$

and

$$
(34)\qquad A_k = 1, \qquad B_k = -2C'_k, \qquad C_k = 2B'_k.
$$

The relation $A_k = 1$ is equivalent to Theorem 5. The other two relations in (34) may be verified by means of (26) and (33). They show that our transformed system of differential equations is in its canonical form.

The invariant $\mathfrak{D}$, whose vanishing characterizes the original conjugate system as a harmonic one, reduces to

$$
(35)\qquad \mathfrak{D} = D + B'_v - C'_u + 3(B'^2 - C'^2),
$$

since we are assuming $A = 1$. Let us denote by $\mathfrak{D}_k$ the corresponding invariant for any conjugate system of the pencil, so that

$$
(36)\qquad \mathfrak{D}_k = D_k + (B'_k)_{\bar v} - (C'_k)_{\bar u} + 3(B'_k)^2 - 3(C'_k)^2.
$$

From (33) and (26) we find

$$
(37)\qquad
\begin{aligned}
(1+k^2)^2B'_k &= (1-3k^2)B' + (k^3-3k)C',\\
(1+k^2)^2C'_k &= -(k^3-3k)B' + (1-3k^2)C',\\
(1+k^2)^2D_k &= (1-k^2)D - 4kD'.
\end{aligned}
$$

If θ is any function of u and v, we find from (31),

$$
(38)\qquad \theta_{\bar u} = \frac{1}{1+k^2}(\theta_u - k\theta_v), \qquad \theta_{\bar v} = \frac{1}{1+k^2}(k\theta_u + \theta_v).
$$

Consequently we obtain the formulæ

$$
\begin{aligned}
(1+k^2)^3(C'_k)_{\bar u} &= -(k^3-3k)(B'_u - kB'_v) + (1-3k^2)(C'_u - kC'_v),\\
(1+k^2)^3(B'_k)_{\bar v} &= (1-3k^2)(kB'_u + B'_v) + (k^3-3k)(kC'_u + C'_v),
\end{aligned}
$$

whence

$$
(39)\qquad
\begin{aligned}
(1+k^2)^4\mathfrak{D}_k = (1+k^2)^2[(1-k^2)D - 4kD' - 2k(B'_u + C'_v)&\\
+ (1-k^2)(B'_v - C'_u)]&\\
+ 3(1-k^2)(1-14k^2+k^4)(B'^2 - C'^2)&\\
+ 12k(1-3k^2)(k^2-3)B'C'.&
\end{aligned}
$$

For $k = 0$, $\mathfrak{D}_k$ reduces to (35), and for $k = 1$ to

$$
(40)\qquad \mathfrak{D}_1 = -D' - \tfrac{1}{2}(B'_u + C'_v) + 3B'C'.
$$

Let us assume that $\mathfrak{D}$ and $\mathfrak{D}_1$ are both equal to zero, so that both the original net and its associate are harmonic, besides being isothermally conjugate. Then $\mathfrak{D}_k$ reduces to the value given by

$$(41)\quad (1 + k^2)^4\mathfrak{D}_k = - 48k(1 - k^2)[k(B'^2 - C'^2) + (1 - k^2)B'C'].$$

If the ratio $B' : C'$ is not a constant, $\mathfrak{D}_k$ can not be equal to zero, for all values of u and v, unless either $k = 0$ or $k = \pm 1$, and these values of k correspond to the original conjugate system and its associate. If the ratio $B' : C'$ *is* a constant which is finite, different from zero or unity, we obtain two values of k, negative reciprocals of each other, and different from 0, ∞, $+ 1$, or $- 1$, by equating to zero the bracketed expression in (41). Thus, there may exist a third net of the pencil, besides the original net and its associate, for which $\mathfrak{D}_k$ is equal to zero. But if $\mathfrak{D}_k$ is equal to zero for more than three distinct nets of the pencil, $\mathfrak{D}_k$ will be equal to zero for all values of k, and B' and C' must vanish. In this case the differential equations of the net reduce to

$$(42)\qquad y_{uu} = y_{vv}, \qquad y_{uv} = 0.$$

Nets of this sort may be described in very simple terms. From equations (42), we conclude

$$y = U(u) + V(v), \qquad U'' = V'' = \tfrac{1}{2}a_1,$$

where $U(u)$ and $V(v)$ are functions of the single variables indicated, and where a_1 is an arbitrary constant. But these equations furnish the following completely integrated expression for y;

$$y = a_1(u^2 + v^2) + a_2u + a_3v + a_4,$$

where a_1, a_2, a_3, a_4 are arbitrary constants. The homogeneous parametric equations of such a net may, therefore, be written in the form

$$y_1 = u^2 + v^2, \qquad y_2 = u, \qquad y_3 = v, \qquad y_4 = 1,$$

whence

$$y_1y_4 - y_2^2 - y_3^2 = 0, \qquad y_2 - uy_4 = 0, \qquad y_3 - vy_4 = 0.$$

Therefore *the sustaining surface of such a net is a quadric. Each of the two component one-parameter families of the net is composed of plane curves (conics), whose planes form a pencil. The axes of these two pencils are conjugate tangents of the quadric surface at one of its points.*

A net with these properties shall be called an *isothermally conjugate quadratic net.* Making use of this terminology we have the following result.

THEOREM 7. *A pencil of isothermally conjugate nets which contains more than three distinct proper harmonic nets is composed entirely of isothermally conjugate quadratic nets.*

We are now in a position to obtain a geometric test for isothermal conjugacy which will be effective in those cases in which theorems 1–4 do not suffice. If a net is isothermally conjugate, every net of its pencil has the property described in theorem 1. If besides, more than three, and

therefore all, of these nets are harmonic, it is an isothermally conjugate quadratic net. Leaving aside this case, we see that the isothermal conjugacy of a net is assured if the property of theorem 1 holds for all of the nets of the pencil and if besides at least one of these nets is known to be non-harmonic.

We may formulate our resulting criterion in the following two theorems.

THEOREM 8. *An isothermally conjugate net possesses the following properties. At every point of the net, the axis tangents, the anti-ray tangents, and the associate conjugate tangents, form three pairs of an involution. Moreover, all of the conjugate nets of the pencil, which is determined by the original net, possess this same property, and no more than three of these nets will be, at the same time, harmonic except in the case of an isothermally conjugate quadratic net.*

THEOREM 9. *Conversely: let there be given a conjugate net such that, at every point of the net, the axis tangents, the anti-ray tangents, and the associate conjugate tangents, form three pairs of an involution. Let all of the conjugate nets of the pencil, determined by the given net, possess the same property, and assume that at least one of the nets of this pencil is not harmonic. Then the original net is isothermally conjugate. If, however, all of the nets of the pencil are harmonic, the original net is an isothermally conjugate net, if and only if it is an isothermally conjugate quadratic net.*

Theorems 8 and 9 together constitute a set of necessary and sufficient conditions for isothermal conjugacy, and these conditions are expressed in purely geometric form. For, according to theorem 2, the question whether a conjugate net is, or is not harmonic, may be decided by examining the double lines of the corresponding involution.

THE UNIVERSITY OF CHICAGO,
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This criterion may be simplified. I have found recently that, if all of the conjugate nets of a pencil are harmonic, they must also be isothermally conjugate. This remark enables us to replace Theorem 9 by

THEOREM 10. *Conversely, let there be given a conjugate net such that, at every point of the net, the axis tangents, the anti-ray tangents, and the associate conjugate tangents, form three pairs of an involution. Let all of the conjugate nets of the pencil, determined by the given net, possess the same property. Then the original net is isothermally conjugate.*

I have also found a second characteristic property of isothermally conjugate nets, which admits of a far simpler statement than that described in theorems 8 and 10. But the detailed presentation of these matters must be left for a future occasion.

OBSERVATIONS WEIGHTED ACCORDING TO ORDER.

By P. J. Daniell.

1. *Introduction.*—When a series of measurements of some quantity are made, two particular quantities require to be calculated expressing respectively the norm and the deviation. For the norm the mean or the median is used while there are three measures of dispersion, the standard or root-mean-square deviation, the mean numerical deviation and the quartile deviation. The question is as to which of these are the more accurate under a general law. Moreover if we choose for our norm the mean or average it appears occasionally profitable to discard one or several extreme measures. Whether, or in what cases, this is legitimate is discussed by Poincaré* but no general conclusions are obtained.

Besides such a discard-average we might invent others in which weights might be assigned to the measures according to their order. In fact the ordinary average or mean, the median, the discard-average, the numerical deviation (from the median, which makes it minimum), and the quartile deviation can all be regarded as calculated by a process in which the measures are multiplied by factors which are functions of order. It is the general purpose of this paper to obtain a formula for the mean square deviation of any such expression. This formula may then be used to measure the relative accuracies of all such expressions.

Certain particular types are discussed and their accuracies calculated in percentages.

Unfortunately the standard deviation is not of the same general type and therefore we add a note on its accuracy. The assumptions made are fairly general. On the one hand the number of observations, n, is supposed large and terms of order higher than $1/n$ are discarded; on the other the probability law assumed is regular and indefinitely differentiable. In our applications to special types, however, we shall only consider cases in which the theoretical distribution is symmetrical, and this for logical reasons. It is useless to compare the relative merits of the various kinds of average, for example, the mean and the median, unless they all tend to coincide when n increases indefinitely. If there is a lack of symmetry both the mean and the median are necessary, or at least valuable, indications of the nature of the distribution. Indeed, in practise, their difference is sometimes regarded as a measure of lack of symmetry.

* Poincaré, "Calcul des Probabilités" (1912), p. 211.

2. *Mathematical Analysis.*—Assume that n measurements $t_1, t_2, \cdots, t_n$ are made and that their magnitudes are in the order of their suffixes, so that

$$t_1 \leqq t_2, \text{ and so on.}$$

Multiply by the factors $f_1, f_2, \cdots, f_n$, so that

$$\bar{t} = \sum_{r=1}^{n} f_r t_r.$$

We desire to find a formula for the mean square deviation of $\bar{t}$ when the measurements, t_r, are subject to some law of probability $p(t)$.

If $\varphi(t_1, \cdots, t_n)$ is some function of the measures considered in their proper order, the average value of φ when $t_1, \cdots, t_n$ vary according to the law of probability will be denoted by Av (φ) to distinguish this from the weighted average, $\bar{t}$, which we obtain for a particular fixed set of values $t_1, t_2, \cdots, t_n$.

Allowing for the possible permutations of the suffixes,

$$\text{Av}(\varphi) = n! \int_{-\infty}^{+\infty} p(t_n)dt_n \int_{-\infty}^{t_n} p(t_{n-1})dt_{n-1} \cdots \int_{-\infty}^{t_2} p(t_1)\varphi(t_1, \cdots, t_n)dt_1.$$

If

$$\int_{-\infty}^{t} p(t)dt = x,$$

let $t = t(x)$; then x varies from 0 to 1, and

$$\text{Av}(\varphi) = n! \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} \psi(x_1, \cdots, x_n)dx_1, \tag{1}$$

where

$$\psi(x_1, \cdots, x_n) = \varphi[t(x_1), \cdots, t(x_n)].$$

We shall make frequent use of the formula

$$\int_0^a dx_p \int_0^{x_p} dx_{p-1} \cdots \int_0^{x_1} f(x)dx = \frac{1}{p!}\int_0^a f(x)(a-x)^p dx. \tag{2}$$

This formula can readily be verified by differentiating with respect to a. A particular case is that in which $f(x) = 1$,

$$\int_0^a dx_p \int_0^{x_p} dx_{p-1} \cdots \int_0^{x_1} dx = \frac{1}{p!}\int_0^a (a-x)^p dx = \frac{1}{(p+1)!}a^{p+1}. \tag{2a}$$

Substituting from (2a) in (1)

$$\text{Av}(1) = n! \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} dx_1 = n! \cdot \frac{1}{n!} \cdot 1^n = 1.$$

This confirms the coefficient $n!$ in the formula (1).

$$\begin{aligned}\text{Av}(t_r) &= n!\int_0^1 dx_n \int_0^{x_n} dx_{n-1}\cdots\int_0^{x_2} t(x_r)dx_1 \\ &= n!\int_0^1 dx_n \int_0^{x_n} dx_{n-1}\cdots\int_0^{x_{r+1}} t(x_r)\frac{x_r^{r-1}}{(r-1)!}dx_r \quad \text{[by 2a]} \\ &= \frac{n!}{(n-r)!\,(r-1)!}\int_0^1 t(x)(1-x)^{n-r}x^{r-1}dx. \qquad (3)\end{aligned}$$

When r, n are large the integrand will have a steep maximum near $x = r/n$. Also

$$\frac{n!}{(n-r)!\,(r-1)!}\int_0^1 x^p(1-x)^{n-r}x^{r-1}dx = \frac{r}{n+1}\cdot\frac{r+1}{n+2}\cdots\frac{r+p-1}{n+p}.$$

Denote $r/(n+1)$ by x_r and neglect terms of order higher than $1/n$.

$$\frac{r+1}{n+2} = x_r + \frac{1}{n}(1-x_r), \qquad \frac{r+2}{n+3} = x_r + \frac{2}{n}(1-x_r), \qquad \text{etc.}$$

$$\begin{aligned}&\frac{n!}{(n-r)!\,(r-1)!}\int_0^1 x^p(1-x)^{n-r}x^{r-1}dx \\ &\qquad = x_r\left[x_r + \frac{1}{n}(1-x_r)\right]\cdots\left[x_r + \frac{p-1}{n}(1-x_r)\right] \\ &\qquad = x_r^p + \frac{p(p-1)}{2n}x_r^{p-1}(1-x_r).\end{aligned}$$

$$\begin{aligned}&\frac{n!}{(n-r)!\,(r-1)!}\int_0^1 (x-x_r)^p(1-x)^{n-r}x^{r-1}dx \\ &\qquad = \sum_{q=0}^{p}\frac{p!}{q!(p-q)!}(-1)^q x_r^p \\ &\qquad + \sum_{q=0}^{p}\frac{p!}{q!(p-q)!}(-1)^q\frac{(p-q)(p-q-1)}{2n}x_r^{p-1}(1-x_r).\end{aligned}$$

Of these two sums the former is 0 unless $p = 0$ and the latter is 0 unless $p = 2$.

$$\begin{aligned}\frac{n!}{(n-r)!\,(r-1)!}\int_0^1 (x-x_r)^p(1-x)^{n-r}x^{r-1}dx &= 0 \qquad (p \neq 0, 2) \\ &= 1 \qquad (p = 0) \\ &= \frac{1}{n}x_r(1-x_r) \qquad (p = 2).\end{aligned}$$

[The reader is reminded that these equations are satisfied only as far as terms of order $1/n$.]

Expand $t(x)$ by a Taylor development near $x = x_r$,

$$t(x) = t(x_r) + (x - x_r)t'(x_r) + \frac{(x - x_r)^2}{2!} t''(x_r) + \cdots.$$

Substitute into (3) and use the formula just obtained, then

$$\text{Av}\ (t_r) = t(x_r) + \frac{1}{2n} x_r(1 - x_r)t''(x_r). \tag{4}$$

By the same reasoning,

$$\begin{aligned} \text{Av}\ (t_r^2) &= t^2(x_r) + \frac{1}{2n} x_r(1 - x_r)[2t(x_r)t''(x_r) + 2\{t'(x_r)\}^2] \\ &= [\text{Av}\ (t_r)]^2 + \frac{1}{n} x_r(1 - x_r)[t'(x_r)]^2. \end{aligned} \tag{5}$$

We next require to calculate Av $(t_r t_s)$ and must agree on order. Suppose $s > r$, then

$$\begin{aligned} \text{Av}\ (t_r t_s) &= n! \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \cdots \int_0^{x_2} t(x_r)t(x_s)dx_1 \\ &= \frac{n!}{(s - r - 1)!\,(r - 1)!} \int_0^1 dx_n \int_0^{x_n} dx_{n-1} \\ &\qquad \cdots \int_0^{x_{s+1}} t(x)dx \int_0^x t(y)(x - y)^{s-r-1}y^{r-1}dy \\ &= \frac{n!}{(n - s)!\,(s - r - 1)!\,(r - 1)!} \\ &\qquad \times \int_0^1 (1 - x)^{n-s}t(x)dx \int_0^x (x - y)^{s-r-1}y^{r-1}t(y)dy. \end{aligned} \tag{6}$$

In this double integral the integrand has a steep maximum near

$$x = s/n, \qquad y = r/n.$$

$$\begin{aligned} &\frac{n!}{(n - s)!\,(s - r - 1)!\,(r - 1)!} \int_0^1 x^p(1 - x)^{n-s}dx \int_0^x y^q(x - y)^{s-r-1}y^{r-1}dy \\ &\quad = \frac{r}{n + 1} \cdot \frac{r + 1}{n + 2} \cdots \frac{r + q - 1}{n + q} \cdot \frac{s + q}{n + q + 1} \cdots \frac{s + q + p - 1}{n + q + p}. \end{aligned}$$

Denote $r/(n + 1)$ by x_r, $s/(n + 1)$ by x_s and expand as far as $1/n$,

$$\frac{r + 1}{n + 2} = x_r + \frac{1}{n}(1 - x_r), \qquad \text{etc.,}$$

$$\frac{s + q}{n + q + 1} = x_s + \frac{q}{n}(1 - x_r), \qquad \text{etc.}$$

$$\frac{n!}{(n-s)!\,(s-r-1)!\,(r-1)!}\int_0^1 x^p(1-x)^{n-s}dx\int_0^x y^q(x-y)^{s-r-1}y^{r-1}dy$$

$$= x_r^q x_s^p + \frac{q(q-1)}{2n}x_r^{q-1}(1-x_r)x_s^p + \frac{p(2q+p-1)}{2n}x_r^q x_s^{p-1}(1-x_s)$$

$$= x_r^q x_s^p + \frac{q(q-1)}{2n}x_r^{q-1}(1-x_r)x_s^p + \frac{p(p-1)}{2n}x_s^{p-1}(1-x_s)x_r^q$$

$$+ \frac{pq}{n}x_r^{q-1}x_s^{p-1}x_r(1-x_s).$$

Using a method similar to that given above,

$$\frac{n!}{(n-s)!\,(s-r-1)!\,(r-1)!}$$

$$\times\int_0^1 (x-x_s)^p(1-x)^{n-s}dx\int_0^x (y-x_r)^q(x-y)^{s-r-1}y^{r-1}dy = 0,$$

except for $p=0,\ q=0;\ p=0,\ q=2;\ p=2;\ q=0;\ p=1,\ q=1.$

In formula (6) expand

$$t(x) = t(x_s) + (x-x_s)t'(x_s) + \cdots, \qquad t(y) = t(x_r) + (y-x_r)t'(x_r) + \cdots,$$

then by similar reasoning as before

$$\begin{aligned}\text{Av }(t_rt_s),\ (s>r) &= t(x_r)t(x_s) + \frac{1}{2n}x_r(1-x_r)t''(x_r)t(x_s)\\ &\quad + \frac{1}{2n}x_s(1-x_s)t(x_r)t''(x_s) + \frac{1}{n}x_r(1-x_s)t'(x_r)t'(x_s) \qquad (7)\\ &= \text{Av }(t_r)\cdot\text{Av }(t_s) + \frac{1}{n}x_r(1-x_s)t'(x_r)t'(x_s).\end{aligned}$$

Now

$$\bar{t} = \sum_{r=1}^{n} f_r t_r$$

$$\text{Av }(\bar{t}) = \sum_{r=1}^{n} f_r\,\text{Av }(t_r),$$

$$\begin{aligned}\text{Av }(\bar{t}^2) &= \sum_{r=1}^{n} f_r^2\,\text{Av }(t_r^2) + 2\sum_{r=1}^{n}\sum_{s=r+1}^{n} f_rf_s\,\text{Av }(t_rt_s) \qquad \text{[By 5, 7].}\\ &= \sum_{r=1}^{n} f_r^2[\text{Av }(t_r)]^2 + 2\sum_{r=1}^{n}\sum_{s=r+1}^{n} f_rf_r\,\text{Av }(t_r)\cdot\text{Av }(t_s)\\ &\quad + \sum_{r=1}^{n} f_r^2\frac{x_r(1-x_r)}{n}[t'(x_r)]^2\\ &\quad + 2\sum_{r=1}^{n}\sum_{s=r+1}^{n} f_rf_r\frac{x_r(1-x_s)}{n}t'(x_r)t'(x_s)\end{aligned}$$

Let $S^2 = n \times x$ mean square deviation of $(\bar{t}) = n[\mathrm{Av}\,(\bar{t^2}) - \mathrm{Av}^2\,(\bar{t})]$. Then let

$$f(x_r) = nf_r, \qquad f(x_r)\Delta x = f(x_r) \cdot \frac{1}{n} = f_r,$$

and replace the double sum by an integral.

$$S^2 = 2\int_0^1 f(x)(1-x)t'(x)dx \int_0^x f(y)yt'(y)dy.$$

Let

$$\varphi(t) = \int_c^t f[x(t)]dt,$$

where c is so chosen that

$$\int_{-\infty}^{+\infty} \varphi(t)p(t)dt = 0. \tag{8}$$

Let

$$\psi(x) = \varphi[t(x)], \qquad \psi'(x) = f(x)t'(x).$$

$$S^2 = 2\int_0^1 (1-x)\psi'(x)dx \int_0^x y\psi'(y)dy.$$

Consider the function

$$F(a, b) = 2\int_a^b (b-x)\psi'(x)dx \int_a^x (y-a)\psi'(y)dy,$$

$$\frac{\partial F}{\partial b} = 2\int_a^b \psi'(x)dx \int_a^x (y-a)\psi'(y)dy,$$

$$\frac{\partial^2 F}{\partial a \partial b} = -2\int_a^b \psi'(x)dx \int_a^x \psi'(y)dy = -[\psi(b) - \psi(a)]^2.$$

Integrating again and since $\partial F/\partial b = 0$, $F = 0$ when $a = b$,

$$\frac{\partial F}{\partial b} = \int_a^b [\psi(b) - \psi(y)]^2 dy, \qquad \frac{dF(0, b)}{db} = \int_0^b [\psi(b) - \psi(y)]^2 dy,$$

$$F(0, b) = \int_0^b dx \int_0^x dy[\psi(x) - \psi(y)]^2,$$

$$\begin{aligned} S^2 &= F(0, 1) \\ &= \int_0^1 dx \int_0^x dy[\psi(x) - \psi(y)]^2 \\ &= \int_{-\infty}^{+\infty} p(t)dt \int_{-\infty}^t p(u)du[\varphi(t) - \varphi(u)]^2. \end{aligned}$$

Interchange the order of integration and also the symbols t, u.

$$S^2 = \int_{-\infty}^{+\infty} p(t)dt \int_t^{\infty} p(u)du[\varphi(u) - \varphi(t)]^2.$$

Combining both forms,

$$S^2 = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[\varphi(t) - \varphi(u)]^2 p(t)p(u)dtdu$$

$$= \int_{-\infty}^{+\infty}\varphi^2(t)p(t)dt - \left[\int_{-\infty}^{+\infty}\varphi(t)p(t)dt\right]^2.$$

But by (8) the last term is 0; then

$$S^2 = \int_{-\infty}^{+\infty}\varphi^2(t)p(t)dt. \tag{9}$$

This is the formula we set out to obtain.

3. *Norm and Deviation.*—For the norm or average $\bar{t} = \sum_r f_r t_r$, with the condition $\sum_{r=1}^{n} f_r = 1$.

Expressing this by the approximate integral and then integrating by parts,

$$\int_{-\infty}^{+\infty} f(t)p(t)dt = 1,$$

$$\int_{-\infty}^{+\infty}[-p'(t)]\varphi(t)dt = 1. \tag{10}$$

The mean is obtained by equal weighting, $f(t) = 1$, $\varphi(t) = t - t_0$, where t_0 is the theoretical average. Then

$$S^2 = \int_{-\infty}^{+\infty}(t - t_0)^2 p(t)dt = \sigma^2.$$

Then the mean square deviation of the mean of n measurements is

$$S^2/n = \sigma^2/n.$$

This particular result is well-known but it confirms our formula (9).

If several groups of measures are to be combined, the average from each group should be multiplied by a factor inversely as the square of the deviation in that group. If then we agree to take the accuracy of the mean as a standard, equal to 1 or 100 per cent., the accuracy of any norm will be measured by the ratio σ^2/S^2.

Definition.—The accuracy of a norm is defined to be σ^2/S^2, where σ^2 is the theoretical square deviation.

In the case of the measure of deviation condition (10) no longer applies but we must suppose the weights f_r chosen so that the average value of the deviation has a fixed value, D.

$$\text{Av}\,(\bar{t}) = \sum_{r=1}^{n} f_r\,\text{Av}\,(t_r) = D.$$

Expressing in integral form, and integrating by parts,

$$D = \int_{-\infty}^{+\infty} tf(t)p(t)dt$$

$$= \int_{-\infty}^{+\infty} \frac{d}{dt}(-tp)\varphi(t)dt \qquad \text{[By (4)]}$$

$$= \int_{-\infty}^{+\infty} [-tp'(t)]\varphi(t)dt - \int_{-\infty}^{+\infty} p(t)\varphi(t)dt.$$

Then, by (8),

$$\int_{-\infty}^{+\infty} [-tp'(t)]\varphi(t)dt = D. \tag{11}$$

For the measure of deviation condition (11) takes the place of (10).

Again if we double the value of D, by doubling f_r, we shall multiply S^2 by 4. A true measure of accuracy will be some multiple of D^2/S^2, and for reasons which appear later we make the

Definition.—The accuracy of a measure of deviation is defined to be $D^2/(2S^2)$, where D is the theoretical average deviation.

Standard Deviation.—The standard deviation may be defined as D where

$$D^2 = \frac{1}{n}\sum_{r=1}^{n} t_r^2 - \frac{1}{n^2}\left(\sum_{r=1}^{n} t_r\right)^2.$$

It is difficult if not impossible to obtain a formula, in the general case, for the average value of D; nevertheless, if the number n is large, the proportional error in D will be small of order $1/\sqrt{n}$. We have the right, therefore, to assume that the proportional error in D will be one half that in D^2. Then if

$$D' = D^2, \qquad S'^2 = n \times \text{mean square deviation of } D',$$

$$\frac{D^2}{2S^2} = \frac{2D'^2}{S'^2}.$$

Choose the origin for t so that

$$\text{Av}\,(t) = \int_{-\infty}^{+\infty} tp(t)dt = 0.$$

Let

$$\int_{-\infty}^{+\infty} t^2 p(t)dt = \sigma^2, \qquad \int_{-\infty}^{+\infty} t^4 p(t)dt = q^4.$$

Then

$$\text{Av}\,(t_r) = \int t_r p(t_r)dt_r \int^{(n+1)} \cdots \int p(t_1)\cdots p(t_n)dt_1\cdots dt_n$$

$$= \int tp(t)dt = 0.$$

$$\text{Av}\,(t_r t_s) = \text{Av}\,(t_r t_s^3) = 0, \qquad \text{Av}\,(t_r^2) = \sigma^2,$$

$$\text{Av}\,(t_r^4) = q^4, \qquad \text{Av}\,(t_r^2 t_s^2) = \sigma^2\cdot\sigma^2 = \sigma^4.$$

But

$$D' = \frac{1}{n}\sum_{r=1}^{n} t_r^2 - \frac{1}{n^2}\left(\sum_{r=1}^{n} t_r\right)^2.$$

The only terms in D' and D'^2 which yield integrals different from 0 will be of the types t_r^2, $t_r^2 t_s^2$, t_r^4.

$$\text{Av}\,(D') = \frac{1}{n}(n\sigma^2) - \frac{1}{n^2}(n\sigma^2) = \frac{n-1}{n}\sigma^2.$$

$$\text{Av}\,(D'^2) = \frac{1}{n^2}\left[nq^4 + 2\frac{n(n-1)}{2}\sigma^4\right] - \frac{2}{n^3}\left[nq^4 + 2\frac{n(n-1)}{2}\sigma^4\right] + \frac{1}{n^4}\left[nq^4 + 6\frac{n(n-1)}{2}\sigma^4\right]$$

$$= \frac{(n-1)^2}{n^3}q^4 + \frac{n-1}{n^3}(n^2 - 2n + 3)\sigma^4.$$

$$S'^2 = n[\text{Av}\,(D'^2) - \{\text{Av}\,(D')\}^2] = \left(\frac{n-1}{n}\right)^2 q^4 - \frac{(n-1)(n-3)}{n^2}\sigma^4.$$

Omitting terms in $1/n$, $1/n^2$,

$$S^2 = \frac{1}{4}D^2 S'^2 \div D'^2 = \frac{D^2}{4}\,\frac{q^4 - \sigma^4}{\sigma^4}. \tag{12}$$

This formula gives the value of n times the mean square deviation of the standard deviation D.

When the theoretical distribution is normal or Gaussian,

$$q^4 = 3\sigma^4, \qquad S^2 = \frac{D^2}{4}\,\frac{3-1}{1} = \frac{D^2}{2}.$$

By the definition, the accuracy will be

$$\frac{D^2}{2S^2} = 1 = 100 \text{ per cent.}$$

This explains the factor 2 which is introduced to make the accuracy of the standard deviation 1 when the law is normal.

It is an interesting fact that the formula (12) proved for the standard deviation is the same as the corresponding value given by (9), when, instead of the mean-root-square deviation, we multiply every measurement by the theoretical value of t corresponding to its order in the series. For then

$$f(t) = \lambda t, \qquad \varphi(t) = \tfrac{1}{2}\lambda t^2 + \text{constant}.$$

This constant is chosen so that $\text{Av}\,(\varphi) = 0$, or

$$\varphi(t) = \tfrac{1}{2}\lambda(t^2 - \sigma^2).$$

Using the condition which led up to (11),

$$\lambda\sigma^2 = D.$$

From (9)

$$S^2 = \frac{\lambda^2}{4}\int_{-\infty}^{+\infty}(t^2-\sigma^2)^2 p(t)dt$$

$$= \frac{\lambda^2}{4}(q^4 - 2\sigma^4 + \sigma^4) = \frac{D^2}{4}\frac{q^4-\sigma^4}{\sigma^4}.$$

4. *Most Accurate Weighting.*—For the norm it is required to make S^2 given by formula (9) a minimum under condition (10). Then

$$\varphi_1(t) = \lambda_1\left[-\frac{p'(t)}{p(t)}\right], \qquad f_1(t) = \lambda_1\frac{d}{dt}\left[-\frac{p'(t)}{p(t)}\right],$$

where from (10)

$$\frac{1}{\lambda_1} = \int_{-\infty}^{+\infty}\frac{p'^2}{p}dt, \qquad S^2 = \lambda_1.$$

Accuracy

$$A_1 = \frac{\sigma^2}{S^2} = \sigma^2\int_{-\infty}^{+\infty}\frac{p'^2}{p}dt.$$

For the deviation, S^2 given by (9) is minimum under condition (11).

$$\varphi_2(t) = \lambda_2\left(-t\frac{p'}{p}-1\right), \qquad f_2(t) = \lambda_2\frac{d}{dt}\left(-t\frac{p'}{p}\right),$$

where from (11)

$$\frac{D}{\lambda_2} = \int_{-\infty}^{+\infty}t^2\frac{p'^2}{p}dt - 1, \qquad S^2 = \lambda_2 D = D^2\frac{\lambda_2}{D}.$$

Accuracy

$$A_2 = \frac{D^2}{2S^2} = \frac{1}{2}\left[\int_{-\infty}^{+\infty}t^2\frac{p'^2}{p}dt - 1\right].$$

For the normal law,

$$-\frac{p'}{p} = \frac{t}{\sigma^2}, \qquad \frac{1}{\lambda_1} = \frac{1}{\sigma^2}, \qquad \frac{D}{\lambda_2} = 2.$$

$$f_1(t) = 1, \qquad f_2(t) = Dt/\sigma^2, \qquad A_1 = 1, \qquad A_2 = 1.$$

Thus for the normal law the most accurate norm is the equal-weighted average and the most accurate deviation that obtained by multiplying each measure by the algebraic theoretical deviation corresponding to its order. As we pointed out before the accuracy of the latter is the same as that of the standard deviation itself.

For the symmetric Pearson law,

$$-\frac{p'}{p} = \frac{2nt}{a^2 \pm t^2},$$

$$f_1(t) = 2\lambda_1 n\frac{a^2 \mp t^2}{(a^2 \pm t^2)^2}, \qquad f_2(t) = \frac{4\lambda_2 n a^2 t}{(a^2 \pm t^2)^2}.$$

If the distribution is supernormal, that is if the number of extreme cases is more than normal the weights for the norm and the weights $\div\, t$ for the deviation should diminish outwards and for the norm should even become negative for large values of t.

On the other hand, if the distribution is subnormal the weights should increase and become infinite at the boundaries $t = \pm a$. In these cases the weighting to be applied is much too complicated to be of any practical value, aside from the impossibility of knowing beforehand the proper values of a and n. However the most important cases are supernormal rather than the reverse and instead of letting the weights diminish according to a complex law we may take equal weights, for the norm, up to a certain point and then discard all measures outside these limits. Such a norm we shall call a discard-average and in practise a certain outer fraction of the measures is discarded. For the deviation we may discard not the outer but the inner fraction. Our next paragraph deals with such special types.

5. *Special Types of Average.—Discard Average.* Let k be the central fraction of the group retained and let t_1 be the solution of

$$2\int_0^{t_1} p(t)dt = k. \tag{13}$$

Then

$$\begin{aligned} f(t) &= \frac{1}{k}, && -t_1 < t < +t_1 \\ &= 0, && \text{otherwise.} \\ \varphi(t) &= \frac{1}{k}t, && 0 < t < t_1 \\ &= \frac{1}{k}t_1, && t > t_1. \end{aligned}$$

By formula (9)

$$S^2 = \frac{1}{k^2}\left[2\int_0^{t_1} t^2p(t)dt + t_1^2 2\int_{t_1}^{\infty} p(t)dt\right].$$

Let α denote the ratio $2\int_0^{t_1} t^2p(t)dt : kt_1^2$,

$$2\int_0^{t_1} t^2p(t)dt = \alpha k t_1^2, \tag{14}$$

$$S^2 = \frac{t_1^2}{k^2}[1 - (1-\alpha)k]. \tag{15}$$

In the case of the median k and t_1 approach 0 together, and

$$S^2 = \frac{1}{4p_0^2}. \tag{15a}$$

In the following discussion for purposes of comparison we shall use the normal Gaussian law and the two extreme Pearson symmetric forms,

$$\text{supernormal,} \qquad p(t) = \frac{2}{\pi}(1 + t^2)^{-2}, \tag{a}$$

$$\text{normal,} \qquad p(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}, \tag{b}$$

$$\text{subnormal,} \qquad p(t) = \frac{3}{4}(1 - t^2), \qquad (t^2 \leqq 1). \tag{c}$$

The accuracy of the median will be, in percentages,

(*a*) 162, (*b*) 63.7, (*c*) 45.

For the quartile-discard average, $k = 1/2$ and the fraction α takes the values,

(*a*) .306, (*b*) .314, (*c*) .323.

Formula (15) becomes

$$S^2 = 2t_1^2(1 + \alpha).$$

The accuracy will be, in percentages,

(*a*) 200,* (*b*) 83.7, (*c*) 63.*

In a supernormal Pearson distribution,

$$p(t) = c(1 + t^2)^{-n}.$$

For the quartile-discard average, when n is large,

$$S^2 = \frac{1.195}{2n}\left[1 + \frac{.851}{n} + \frac{.700}{n^2}\right] = \frac{1.195}{2n - 1.70}.$$

Hence the accuracy of the quartile-discard average will be

$$A = 83.7\frac{2n - 1.70}{2n - 3.00} \text{ per cent.} = 83.7 + \frac{109}{2n - 3} \text{ per cent.}$$

The formula can hardly be used with accuracy when n is as small as 2, but even then it would give the value

$$A = 193.$$

* These values are only rough approximations.

The quartile-discard average will be more accurate than the ordinary mean if

$$2n < 9.3.$$

If σ^2 is average t^2 and q^4 average t^4 then the above condition may be translated into

$$q^4 > 4 \cdot 4\sigma^4,$$

instead of $3\sigma^4$ for a normal law.

Median-Quartile Average.—This average is the mean of the median and the two quartiles,

$$\bar{t} = \frac{1}{3}(Q_1 + M + Q_3), \qquad S^2 = \frac{1}{36}\left(\frac{1}{p_0^2} + \frac{2}{p_0 p_1} + \frac{2}{p_1^2}\right),$$

where $p_0 = p(t_0)$, $p_1 = p(t_1)$ and t_1 is the theoretical quartile deviation.

For a normal law the accuracy is

$$A = 86.0 \text{ per cent.}$$

It appears to be a little more accurate than the quartile-discard average, but we have assumed that the number of observations is large. When the number is small it will be difficult to determine the quartiles exactly, so that, taking everything into consideration, we may say the median-quartile average and the quartile-discard average are about equally accurate.

The most serious objection to the use of any special type of average is that discontinuity is introduced; that is, if the measures are considered as sufficiently normal, none will be discarded; if not, some may be discarded and there will be a finite change in the average. To obviate this difficulty we might use a combination of quartile-discard and ordinary average.

Let p be the weight assigned to the ordinary average.

$q = 1 - p =$ weight assigned to the quartile-discard.

$\sigma^2 =$ mean square deviation.

$N =$ mean numerical deviation.

$P =$ quartile deviation, or probable error.

Then

$$S^2 = p^2\sigma^2 + 2pqP(2N - \tfrac{1}{6}D) + q^2 \tfrac{8}{3} P^2, \tag{16}$$

approximately, assuming average t^2 from 0 to P is $\frac{1}{3}P^2$, average t from 0 to P is $\frac{1}{2}P$. Then we may choose p, q so as to make S^2 minimum.

6. *Special Types of Dispersion Measure. Numerical Deviation.*—In this

case

$$f(t) = +1, \qquad t > 0,$$
$$\quad = -1, \qquad t < 0,$$

$$\varphi(t) = |t| - N,$$

where N is the theoretical mean numerical deviation. Then, by (9),

$$S^2 = \sigma^2 - N^2.$$

By the definition, succeeding formula 11, the accuracy will be

$$\frac{N^2}{2S^2} = \frac{N^2}{2(\sigma^2 - N^2)}\,.$$

In the case of our three Pearson types (*a*) supernormal, (*b*) normal, (*c*) subnormal, the accuracy, in percentages, will be

(*a*) 34, (*b*) 87.6, (*c*) 118.

Discard Deviation.—In this case we discard the inner portion and then use the mean numerical deviation of the remainder. Under a normal law, if the portion between the quartiles, that is the central half, is discarded the accuracy is 96.3 per cent. Hence this is practically as accurate as the standard deviation and may, in some cases, be more rapidly found as it is a numerical mean and the calculations are made for half the measures only.

Quartile Deviation, etc.—If t is the theoretical deviation the accuracy will be

$$8[tp(t)]^2.$$

For the quartile deviation this is 36.7 per cent.

It will be a maximum when $tp(t)$ is maximum and for a normal law this is given by $t = \sigma$. The values $t = \pm\sigma$ practically divide the whole range of measures into two thirds within and one third without.

We may call this the sextile deviation, remembering that it is the outermost sextiles only which are used. Then the accuracy becomes 46.8 per cent. Furthermore it is much easier to find the corresponding standard deviation σ in this case, for theoretically with a normal law

$$\sigma = \left(1 + \frac{1}{30}\right) \text{outer-sextile deviation.}$$

We might also call this the semi-probable error, that is, such that the chances of exceeding it are just one half the chances of not exceeding.

A table is added of the accuracies of the various types when the law is assumed to be normal.

Norms:		
Median	63.7	per cent.
Quartile average $(Q_1 + M + Q_3)/_3$	86.0	" "
Quartile discard average (outer quartiles discarded)	83.7	" "
Mean	100	" "
Dispersion:		
Quartile deviation	36.7	" "
Outer sextile deviation	46.8	" "
Numerical deviation	87.6	" "
Discard deviation (inner quartiles discarded)	96.3	" "
Standard deviation	100	" "

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SOME DETERMINANT EXPANSIONS.*

By Lepine Hall Rice.

§ 1. There is a recent paper by Sir Thomas Muir† which presents an important general theorem upon the expansion of a determinant. Muir states the theorem summarily as follows:

A determinant can be expressed in terms of minors drawn from four mutually exclusive arrays, two of which are coaxial and complementary to one another.

The discussion leading up to this statement involves bordered determinants. But without reference to such determinants, by following a line of thought suggested by matter contained in §§ 11 and 12 of Muir's paper, it will be found possible not only to prove the theorem in a very simple manner but also to obtain several progressively broader results.

We need first, however, to state the theorem more specifically and in such a manner as to prepare for its extension. With respect to the four arrays it is to be noted that they may be marked out by drawing two lines, one horizontal and the other vertical, across the matrix of the determinant Δ, intersecting on the main diagonal line between two elements thereof; an arrangement which we shall denote as follows:

$$\|\Delta\| \equiv \begin{Vmatrix} A & B \\ C & D \end{Vmatrix};$$

$$A \equiv \begin{Vmatrix} a_{11} \cdots a_{1p} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{p1} \cdots a_{pp} \end{Vmatrix}, \qquad B \equiv \begin{Vmatrix} b_{11} \cdots b_{1q} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ b_{p1} \cdots b_{pq} \end{Vmatrix},$$

$$C \equiv \begin{Vmatrix} c_{11} \cdots c_{1p} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ c_{q1} \cdots c_{qp} \end{Vmatrix}, \qquad D \equiv \begin{Vmatrix} d_{11} \cdots d_{1q} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ d_{q1} \cdots d_{qq} \end{Vmatrix}, \qquad p + q = n.$$

Further we must particularize with respect to the words "expressed in terms of minors drawn from" the arrays. It is well known that if a set of minors of a determinant Δ, such that their row numbers together are the row numbers of Δ and their column numbers the column numbers of Δ, be taken as the factors of a product to which is prefixed the sign of that term of Δ whose elements are the elements of the main diagonal terms of the minors, this product is identical with the sum of a certain number

* Presented to the American Mathematical Society, Sept. 2, 1919.

† Note on the representation of the expansion of a bordered determinant, by Sir Thomas Muir, LL.D., *Mess. Math.*, No. 566, Vol. xlviii., June, 1918.

of terms of Δ. As elsewhere,* we shall call any two minors of a determinant, which are susceptible of entering into such a set, *conjunctive* minors, and the whole a set of *perjunctive* minors or a *perjunct*; it is a *signed* perjunct if the specified sign is prefixed. When all the minors are of the first order or simply elements of Δ, we have a *transversal* of Δ; if signed, a *term*. The meaning of the phrase in the theorem then is that every possible signed perjunct is to be formed whose minors are four in number and lie one in each of the four arrays. It is understood that any one or more of these minors may be of order zero, with the value 1. And throughout this paper a perjunct will be understood to admit minors of order zero. Let a minor lying wholly in A be called an a-minor, and so for B, C, and D.

We are now ready to restate Muir's theorem as

THEOREM A. *If the matrix of any determinant Δ be partitioned, by a horizontal and a vertical line intersecting on the main diagonal line, into four arrays, A, B, C, and D, then Δ can be expanded as the sum of all signed perjuncts composed of an a-minor, a b-minor, a c-minor, and a d-minor.*

§ 2. In the proof of this theorem which we shall now give, and in further proofs in this paper, our line of thought concerns the individual terms of the determinant to be expanded, in their relation to the specified arrays into which the matrix of the determinant is divided.

Consider then any term of Δ. Separate its elements into those lying in A, those lying in B, those lying in C, and those lying in D. The four groups of elements determine by their row and column numbers four minors lying respectively in the four arrays and forming a perjunct which is evidently the only perjunct of four minors lying in the four arrays which contains this term.

Thus the sum of perjuncts specified in the theorem contains nothing but terms of Δ and contains every term once and only once. It is therefore an expansion of Δ.

§ 3. We are immediately led to give the theorem additional breadth by removing the condition that the horizontal and vertical lines must intersect on the main diagonal line, for the proof does not hang upon that condition; and we have

THEOREM 1. *If the matrix of any determinant Δ be partitioned by a horizontal and a vertical line into four arrays, A, B, C, and D, then Δ can be expanded as the sum of all signed perjuncts composed of an a-minor, a b-minor, a c-minor, and a d-minor.*

§ 4. To illustrate this expansion let us take a determinant of order 7, partitioned thus:

$$\Delta \equiv \begin{vmatrix} \| a_{35} \| & \| b_{32} \| \\ \| c_{45} \| & \| d_{42} \| \end{vmatrix}.$$

* P-way determinants, with an application to transvectants, AMERICAN JOURNAL OF MATHEMATICS, Vol. XL, No. 3, July, 1918, p. 242. Cited herein as P-way dets.

(i) Take an a-minor of order 3, a d-minor of order 2, and the b-minor and c-minor determined thereby, the b-minor being of course of order zero while the c-minor is of order 2; example, $a_{\substack{123\\123}} c_{\substack{12\\45}} d_{\substack{34\\12}}$. (ii) Take an a-minor of order 2, a d-minor of order 1 (an element), and the b-element and c-minor of order 3 determined thereby; example, $-\,a_{\substack{12\\12}} b_{\substack{3\\1}} c_{\substack{123\\345}} d_{\substack{4\\2}}$. (iii) Finally, take an a-element (the d-minor now being of order zero), and the b-minor of order 2 and c-minor of order 4 determined by the a-element; example, $a_{\substack{1\\1}} b_{\substack{23\\12}} c_{\substack{1234\\2345}}$. As a check, we may count up in the result the terms of Δ: $\binom{5}{3}\binom{4}{2}3\,!\,2\,!\,2\,! + \binom{5}{2}\binom{3}{2}\binom{4}{1}\binom{2}{1}2\,!\,3\,! + \binom{5}{1}\binom{3}{1}2\,!\,4\,! = 7\,!$.

This procedure is applicable generally. We start by forming all possible perjuncts consisting of one of the largest a-minors and one of the largest d-minors, together with the b-minor and c-minor determined thereby; next we form all possible perjuncts with the a-minor and d-minor one less in order, and the b-minor and c-minor one greater; and so we continue, until the a-minor or the d-minor becomes of order zero.

§ 5. The next generalization consists in removing altogether the restrictions on the manner of partitioning the matrix of Δ into rectangular arrays. Let us call a rectangular array which is a part of the matrix of Δ a *panel* of Δ. Panels may be of any number and each may be of any dimensions so long as all fit together into the square matrix. With slight and obvious changes the former proof covers this more inclusive case, and we have

THEOREM 2. *If the matrix of a determinant Δ be partitioned into panels in any manner, then Δ can be expanded as the sum of all signed perjuncts composed of one minor from each panel.*

§ 6. A theorem was announced by Albeggiani in 1875 in a paper entitled *Sviluppo di un determinante ad elementi polinomi*,* which interests us here for three reasons. First, it can be proved in the manner of § 2 with directness and brevity. Secondly, it can be utilized to establish Theorem 2. And thirdly, it can be generalized from two dimensions to three or more.

As Albeggiani himself pointed out, this theorem applies to any determinant whatever, for polynomial elements can be made out of monomial elements *ad libitum*, either by breaking up the monomial elements or by annexing zero terms. Consider then the general determinant $\Delta = |a_{1n}|$, and put

$$a_{ij} = h_{ij}^{(1)} + h_{ij}^{(2)} + \cdots + h_{ij}^{(r)}.$$

Set up the r determinants

$$\Delta^{(k)} \equiv \begin{vmatrix} h_{11}^{(k)} \cdots h_{1n}^{(k)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ h_{n1}^{(k)} \cdots h_{nn}^{(k)} \end{vmatrix}, \qquad k = 1, 2, \cdots, r.$$

* *Giorn. di Batt.*, Vol. 13, p. 1.

Form what we may call a *mixed perjunct* by taking one minor from $\Delta^{(1)}$, a second minor, conjunctive in position, from $\Delta^{(2)}$, and so on to $\Delta^{(r)}$, and prefix the sign determined precisely as it would be determined if all the minors came from one determinant. Then we may state Albeggiani's theorem as follows:

If Δ be any determinant, the sum of all the signed mixed perjuncts from r determinants so formed that the sum of their matrices is the matrix of Δ, is an expansion of Δ.

To prove the theorem, consider any term of Δ. This a-term yields r^n monomials each the product of n h's, which may be called h-terms of Δ. Now obviously we can think of expanding Δ directly into its h-terms without first forming the a-terms. And from that point of view it is clear that any given h-term is to be found in one and only one mixed perjunct. For, separate the elements of this h-term into the $h^{(1)}$'s, the $h^{(2)}$'s, and so on. These groups determine just one mixed perjunct containing this term; and therefore, as the perjuncts contain nothing but h-terms of Δ, we have an expansion of Δ.

§ 7. In order to prove Theorem 2 by means of Albeggiani's theorem, we form $\Delta^{(1)}, \Delta^{(2)}, \cdots, \Delta^{(r)}$ by writing into r blank matrices the r panels of Δ, each in its proper place, and then filling up each matrix with zeros. All minors of $\Delta^{(1)}$ vanish except those lying in the first panel, all minors of $\Delta^{(2)}$ except those lying in the second panel, and so on. The mixed perjuncts which survive are identical with the perjuncts of Δ specified in Theorem 2.

§ 8. Let us next extend Albeggiani's theorem to cubic or 3-way determinants, preparatory to an extension to p-way determinants. Let*

$$\Delta = |a_{\breve{\alpha}\hat{\beta}\hat{\gamma}}|_n; \qquad a_{\alpha\beta\gamma} = h^{(1)}_{\alpha\beta\gamma} + h^{(2)}_{\alpha\beta\gamma} + \cdots + h^{(r)}_{\alpha\beta\gamma}.$$

Set up the r determinants

$$\Delta^{(k)} = |h^{(k)}_{\breve{\alpha}\hat{\beta}\hat{\gamma}}|_n; \qquad k = 1, 2, \cdots, r.$$

Then we have

THEOREM 3. *If Δ be any 3-way determinant, the sum of all the signed mixed perjuncts from r determinants of the same signancy as Δ, the sum of whose matrices is the matrix of Δ, is an expansion of Δ.*

The proof follows that of § 6 very closely, the introduction of the nonsignant third index giving no trouble.

Defining a *block* as a 3-way rectangular matrix forming a part of the matrix of Δ, we have the

COROLLARY. *If the matrix of a 3-way determinant Δ be partitioned into blocks in any manner, then Δ can be expanded as the sum of all signed perjuncts composed of one minor from each block.*

* For the notation, see *P*-way dets., §§ 5, 6.

In particular, the blocks may be formed by three mutually perpendicular planes passed through the matrix. The types of perjuncts become here much more numerous than in the case of a 2-way determinant under Theorem 1. In any speical determinant there may be blocks the character of whose elements will simplify the application of the Corollary.

§ 9. Finally, consider the general p-way determinant

$$\Delta = |a_{\alpha\beta\cdots\kappa\lambda}|_n^{(p)},$$

in which any or all of the indices may be signant or nonsignant. Put

$$a_{\alpha\beta\cdots\kappa\lambda} = h^{(1)}_{\alpha\beta\cdots\kappa\lambda} + h^{(2)}_{\alpha\beta\cdots\kappa\lambda} + \cdots + h^{(r)}_{\alpha\beta\cdots\kappa\lambda},$$

and form r determinants of the same signancy as Δ:

$$\Delta^{(k)} = |h^{(k)}_{\alpha\beta\cdots\kappa\lambda}|_n^{(p)}, \qquad k = 1, 2, \cdots, r.$$

THEOREM 4. *If Δ be any p-way determinant, the sum of all the signed mixed perjuncts from r determinants of the same signancy as Δ, the sum of whose matrices is the matrix of Δ, is an expansion of Δ.*

Proof. First, to show that a signed perjunct consists of a certain number of terms of Δ. That the perjunct consists of transversals of Δ, is clear. It is now to the correspondence of signs that we must look. And it will be perceived that this point is really settled by the known correspondence in the case of a 2-way determinant. For, the argument in that case considers, first, row numbers, next, column numbers, treating both sets in the same way and combining the results. Here we have simply to apply the same argument to each signant index in turn, and to combine the results by taking the product of the signs of the signant ranges.

Secondly, to find any given h-term of Δ in one and only one mixed perjunct. We group the $h^{(1)}$'s, the $h^{(2)}$'s, and so on. Previous reasoning is here follc ved and the result readily comes, completing the proof.

Extend the definition of a *block* to p dimensions: it is to consist of all those elements for which α has a value found among a fixed set of values $\alpha_1, \alpha_2, \cdots, \alpha_{b_1}$; β, a value found among a set of values $\beta_1, \beta_2, \cdots, \beta_{b_2}$; and so on. The locant of the block is thus

$$\left\{\begin{matrix} \alpha_1\alpha_2 \cdots \alpha_{b_1} \\ \beta_1\beta_2 \cdots \beta_{b_2} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \lambda_1\lambda_2 \cdots \lambda_{b_p} \end{matrix}\right\}.$$

We shall then evidently have, under Theorem 4, the

COROLLARY. *If the matrix of a p-way determinant Δ be partitioned into blocks in any manner, then Δ can be expanded as the sum of all signed perjuncts composed of one minor from each block.*

§ 10. It is important to note that all of the foregoing results apply to permanents as well as to determinants, since the reasoning in no case depends—as does, for instance, the reasoning which establishes the multiplication theorem—upon the vanishing of certain aggregates of terms.

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A GENERAL IMPLICIT FUNCTION THEOREM WITH AN APPLICATION TO PROBLEMS OF RELATIVE MINIMA.

By K. W. Lamson.

Goursat has given a proof of the existence of a system of solutions of the equations

$$y_i = F_i(y_1, \cdots, y_n;\ z_1, \cdots, z_m) \qquad (i = 1, 2, \cdots, n), \tag{1}$$

where the functions F_i reduce to $y_i^{(0)}$ for $y = y^{(0)}$, $z = z^{(0)}$, and their difference from $y_i^{(0)}$ is of an order higher than the first in the variables y. He has further shown how, under certain conditions, the following system

$$G_i(y_1, \cdots, y_n;\ z_1, \cdots, z_m) = 0 \qquad (i = 1, 2, \cdots, n), \tag{2}$$

can be reduced to the form (1). A system of equations of type (2) arises in the theory of relative extrema of functions of a finite number of variables (referred to as theory I).

Equations (1) and (2) suggest the following problem of implicit functions in the theory of Functions of Lines. Let x, ξ be variables on the continuous range ab, and consider a functional operation $F[y(x), z(x);\ \xi]$ such that to a pair of functions $y(x)$, $z(x)$ and number ξ on ab corresponds a unique real number. Further suppose that $F[y(x), z(x);\ \xi]$ reduces to y_0 when $y = y_0$, $z = z_0$, and that its difference from y_0 is of an order higher than the first, with a suitable definition of order of difference. The subscript i, thought of as a variable with the discrete range $1, 2, \cdots n$, or $1, 2, \cdots m$, has been replaced by the variable ξ with the continuous range ab. The functions $y(x)$, $z(x)$ take the place of the sets of numbers y_i, z_i. To equations (1) and (2) correspond

$$y(\xi) = F[y(x), z(x);\ \xi], \tag{3}$$

$$G[y(x), z(x);\ \xi] = 0. \tag{4}$$

Fréchet uses the term "fonctionelle" for F or G, when ξ is fixed, and the term "functional" has come into use as the English equivalent. For equation (3), Volterra* has suggested an existence proof analogous to that of Goursat for equation (1). An instance of equation (4) occurs in the Calculus of Variations in the case of problems in the plane (referred to as theory II).

The first purpose of this paper is to give an existence proof for equations

* *Leçons sur les Fonctions des Lignes*, p. 71.

which include as special cases equations (1), (2), (3) and (4). Equations (3) and (4), although suggested by (1) and (2) are not generalizations of them in the sense of including them as special cases. The general theory is to include also the systems of equations of type (4) appearing in the space problems of the Calculus of Variations (referred to as theory III). The existence theorems used in the theories I, II and III have similarities in hypothesis, proof and conclusion. In I a solution consists of a set of numbers y_i, a function of the variable i, with the range $i = 1, 2, \cdots n$; in II the solution is a function $y(x)$ of the continuous variable x, with the range $a \leqq x \leqq b$, and in III it is a function $y_i(x)$ of the composite variable (i, x) with the composite ranges $i = 1, 2, \cdots n$, $a \leqq x \leqq b$. The difference in the three theories lies in the difference in the range of the independent variable. Any general theory which includes the three as special cases will introduce a range which will specialize to the three just mentioned. For two reasons it has seemed best not to attempt to abstract common properties from these ranges, but to introduce the general range* of E. H. MOORE, not defined and on which no postulates bear explicitly. In the first place the dissimilarities make it hard to find useful common properties, and in the second place, the general theory is not to exclude problems involving double integrals or combinations of integrals and sums. The general range is a set $\mathfrak{P}$ of elements p, and the functions to be considered are such that to each p corresponds a real number $y(p)$.

Replace the i of equations (1) and (2) and the x of (3) and (4) by p. This leads to the equations

$$y(p) = F[y(q), z(q);\ p], \tag{5}$$

$$G[y(q), z(q);\ p] = 0, \tag{6}$$

where q has the range $\mathfrak{P}$ and where, by means of F and G, to each p and pair of functions y and z in a certain class $\mathfrak{M}$ of functions there corresponds a unique real number.

In § 1 below the basis and postulates for the solution of equations (5) and a special form of (6) are set down. In §§ 2, 3 are lemmas leading to the solution of (5) and to the reduction of (6) to the form (5). The last section of the paper contains an application to the problem of Lagrange in the Calculus of Variations.

§ 1. *The Basis.*

The independent variable of the theory has the general range $\mathfrak{P}$. An element of $\mathfrak{P}$ will be denoted by one of the letters p, q. The functions entering the theory belong to a class $\mathfrak{M}$, whose elements are real single-valued functions $y(p)$ or $z(p)$. In theory I the class $\mathfrak{M}$ is the set of n-

* Bolza, *Bulletin of the American Mathematical Society*, Vol. 16 (1910), p. 403; also *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Vol. 23 (1914), p. 251.

partite numbers or of points in n-space. In theories II and III $\mathfrak{M}$ is the class of functions or curves in the plane and in $(n+1)$-space respectively, continuous with their first derivatives. To each element y of $\mathfrak{M}$ corresponds a positive or zero number, the "modulus" of y, which will be denoted by $\|y\|$. In theory I the modulus is interpreted as the largest of the numbers y_i, or as the distance of the point $(y_1, \cdots, y_n)$ from the origin. In theories II and III the modulus is interpreted as the number defining a neighborhood of the first order, namely the maximum absolute value of the functional value and of the derivative. In the general theory the modulus is not defined and is subject to postulates. These postulates and those on $\mathfrak{M}$ will be shown in § 4 to be satisfied in the case of the Lagrange problem.

Postulate 1. $\mathfrak{M}$ is linear, that is, contains all functions of the form $c_1y_1 + c_2y_2$, where c_1 and c_2 are real numbers, provided y_1 and y_2 are themselves in $\mathfrak{M}$.

Postulate 2. $\|y_1 + y_2\| \leqq \|y_1\| + \|y_2\|$.

Postulate 3. $\|cy\| = |c|\,\|y\|$, for every real number c.

Postulate 4. If $\|y\| = 0$, then $y(p) = 0$ for every p.

THEOREM 1. *If $\{y_i\}$ and $\{y'_i\}$ are sequences, and y and y' are functions, such that* $\lim_{i=\infty} \|y - y_i\| = \lim_{i=\infty} \|y' - y'_i\| = 0$; *and if* $\lim_{i\to\infty} \|y_i - y'_i\| < b$, *then* $\|y - y'\| \leqq b$.

This theorem follows at once from the preceding postulates.

Definition. The sequence $\{y_i\}$ is defined to be a Cauchy sequence if

$$\lim_{i=\infty,\, j=\infty} \|y_i - y_j\| = 0.$$

The sequence $\{y_i\}$ is said to have a limit y if $\lim_{i=\infty} \|y - y_i\| = 0$. The uniqueness of this limit is a result of postulates 2, 3, 4.

Postulate 5. For every Cauchy sequence in $\mathfrak{M}$ there exists a function in $\mathfrak{M}$ which is the limit of this sequence.

Definition. The symbol $(\bar{y})_a$ denotes the totality of functions y of $\mathfrak{M}$ such that $\|y - \bar{y}\| < a$.

Consider $F[y_1, \cdots, y_\kappa; p]$ real and single-valued for y_i in $(\bar{y}_i)_a$ $(i = 1, \cdots \kappa)$ and p in $\mathfrak{P}$, and such that when $y_1, \cdots, y_\kappa$ are fixed the resulting function of p is in $\mathfrak{M}$.

Definition. The functional F is continuous at a set of arguments $(y'_1, \cdots, y'_\kappa)$ if for every ϵ there exists a δ such that

$$\|F[y_1, \cdots, y_\kappa; p] - F[y'_1, \cdots, y'_\kappa; p]\| < \epsilon$$

whenever y_i is in $(y'_i)_\delta$.

§ 2. *Solution of the equation* $y(p) = F[y, z; p]$.

The proof of the existence of a solution, $y(p)$, of the equation

$$y(p) = F[y, z; p] \tag{5}$$

is similar to that given by Goursat,* who used the method of successive approximations to treat equation (1). Let y_0 and z_0 be two functions of the class $\mathfrak{M}$. The functional $F[y, z; p]$ is supposed to be real and single-valued for all elements $(y, z; p)$ for which y is in a neighborhood $(y_0)_a$, z in $(z_0)_a$, and p in $\mathfrak{P}$, and to have the property that when y and z are fixed in its range of definition the resulting function of p is also in $\mathfrak{M}$. It has further the properties

(1) $F[y_0, z_0; p] = y_0(p)$ for every p in $\mathfrak{P}$;

(2) it is continuous in y and z at each element y', z', in its range of definition;

(3) there exists a constant $0 < K < 1$ such that

$$\| F[y_1, z; p] - F[y_2, z; p] \| < K \| y_1 - y_2 \|$$

whenever (y_1, z) and (y_2, z) are in the range for which F is defined. This condition will be referred to as the Lipschitz condition.

Define a sequence of successive approximations by the equations

$$y_1 = F[y_0, z; p] \tag{7}$$

$$y_{i+1} = F[y_i, z; p] \qquad (i = 1, 2, 3, \cdots), \tag{8}$$

which is possible whenever every y_i is in $(y_0)_a$. It will first be shown that a neighborhood $(z_0)_{a_1}$ with $a_1 \leqq a$ can be chosen so that the elements of the sequence are well defined whenever z is in $(z_0)_{a_1}$.

LEMMA 1. *There exists a positive constant $a_1 \leqq a$ such that for z in $(z_0)_{a_1}$, and for every i, y_i is in $(y_0)_a$.*

To prove this, use the continuity of F in z, and choose $a_1 \leqq a$, so that

$$\| y_1 - y_0 \| = \| F[y_0, z; p] - F[y_0, z_0; p] \| < a(1 - K).$$

From the Lipschitz condition, if y is in $(y_0)_a$,

$$\| F[y, z; p] - F[y_0, z; p] \| < K \| y - y_0 \|.$$

From the addition of $\| F[y_0, z; p] - y_0 \|$ to both sides, and from Postulate 2, follows

$$\| F[y, z; p] - y_0 \| < K \| y - y_0 \| + a(1 - K). \tag{9}$$

In particular, putting $y = y_1$, this becomes

$$\| y_2 - y_0 \| < K \| y_1 - y_0 \| + a(1 - K) < a.$$

To complete the induction proof, assume $\| y_i - y_0 \| < a$, and put y_i in (9).

* Goursat, *Bulletin de la Société Mathématique de France*, Vol. 31 (1903), p. 184. Bliss, *Princeton Colloquium Lectures*, p. 8.

LEMMA 2. *The sequence* $\{y_i\}$ *is a Cauchy sequence and its limit,* y, *(Postulate 5) is in* $(y_0)_a$.

To prove this, the convergence of the series $\sum_i ||y_{i+1} - y_i||$ is first shown, by using $\sum_i K^i a$ as a dominating series. From the definition of y_2 and y_1, and from the Lipschitz condition,

$$||y_2 - y_1|| = ||F[y_1, z; p] - F[y_0, z; p]|| < K||y_1 - y_0|| < Ka.$$

To complete the induction proof, assume

$$||y_{i+1} - y_i|| < K^i a,$$

and apply the Lipschitz condition to $||y_{i+2} - y_{i+1}||$.

The convergence of $\Sigma||y_{i+1} - y_i||$, and Postulate 2 imply that the sequence $\{y_i\}$ is a Cauchy sequence. From Theorem 1 it follows that the limit y of $\{y_i\}$ is in $(y_0)_a$.

LEMMA 3. *The equation* (5) *is satisfied by the limit* y *of Lemma* 2.

For from the definition of y_i, and from Lemma 2,

$$\lim_{i=\infty} ||y - y_i|| = \lim_{i=\infty} ||y - F[y_i, z; p]|| = 0. \tag{10}$$

From the continuity of F,

$$\lim_{i=\infty} ||F[y_i, z; p] - F[y, z; p]|| = 0, \tag{10a}$$

and from the addition of (10) and (10a), and the application of Postulates 2 and 4,

$$y = F[y, z; p].$$

LEMMA 4. *The solution* y *of equation* (5) *described in the preceding lemmas is the only one in* $(y_0)_a$ *corresponding to a* z *in* $(z_0)_{a_1}$.

For the proof, assume two solutions, and apply the Lipschitz condition to their difference, using Postulate 4.

LEMMA 5. *As a functional of* z, y *is continuous in the neighborhood* $(z_0)_{a_1}$.

It is necessary to show that if $||z - z'||$ is small, then $||y - y'||$ is small, where y and y' are the solutions corresponding to z and z' respectively.

From Postulate 2,

$$\begin{aligned} ||y - y'|| &= ||F[y, z; p] - F[y', z'; p]|| \\ &\leqq ||F[y, z; p] - F[y', z; p]|| + ||F[y', z; p] - F[y', z'; p]|| \\ &\leqq K||y - y'|| + ||F[y', z; p] - F[y', z'; p]||. \end{aligned}$$

From the continuity in z, the last term can be made less than an ϵ as required, whence

$$||y - y'|| < \frac{\epsilon}{1 - K}.$$

The results of this section may be summed up in the following

THEOREM 2. *When $F[y, z; p]$ has the solution $(y_0, z_0; p)$ and the properties described at the beginning of this section for elements $(y, z; p)$ with y in $(y_0)_a$, z in $(z_0)_{a_1}$ and p in $\mathfrak{P}$, there exists a constant $a_1 \leqq a$ such that the equation*

$$y = F[y, z; p]$$

has one and only one solution $y = Y[z; p]$ for each z in the neighborhood $(z_0)_{a_1}$. The functional $Y[z; p]$ so defined has the value $y = y_0$ for $z = z_0$ and is continuous at $z = z_0$.

§ 3. *The equation $G[y; p] = z(p)$.*

In order to transform equation (2) to the form (1), Goursat* assumes first that the derivatives $\partial G_i/\partial y_j$ exist and are continuous, and second that the functional determinant is different from zero for those values of y_i and z_i for which the G_i vanish. The equation (6) will be taken in the less general form,

$$G[y; p] = z(p), \tag{11}$$

which is to be solved for y, given that

$$G[y_0; p] = z_0(p).$$

The equation (11) will be transformed to the form (5) treated in the preceding section, by a procedure following that of Goursat. Before prescribing the properties of the functional G it will be useful to describe those of a functional $A[y_1, y_2, \eta; p]$ which will be called a difference function for reasons which will presently appear. At each element $(y_1, y_2, \eta; p)$ with y_1 and y_2 in $(y_0)_a$, η in $\mathfrak{M}$, and p in $\mathfrak{P}$ the functional A has a single real value, and when the first three of its arguments are fixed defines a function of the class $\mathfrak{M}$. It has furthermore the following properties:

(1) it is linear in η, that is,

$$A[c_1\eta_1 + c_2\eta_2] = c_1A[\eta_1] + c_2A[\eta_2]$$

where η_1 and η_2 are functions of the class $\mathfrak{M}$ and c_1 and c_2 are constants. The three arguments other than η are suppressed for the moment in this equation;

(2) There exists a constant M such that

$$\| A[y_1, y_2, \eta; p] \| \leqq M \| \eta \|$$

whenever $(y_1, y_2, \eta; p)$ is in the set for which A is defined;†

(3) the functional A is uniformly continuous in (y_1, y_2) at (y_0, y_0) with

* Loc. cit., p. 191.

† Riesz, *Annales Scientifique de L'École Normale Supérieure*, 3me Série, Vol. 31 (1914), p. 10.

respect to the set of admissible arguments η for which $||\eta|| = 1$, that is for every given ϵ there exists a δ such that

$$|| A[y_1, y_2, \eta; p] - A[y_0, y_0, \eta; p] || < \epsilon$$

whenever y_1 and y_2 are in $(y_0)_\delta$, η is in $\mathfrak{M}$.

The functional $G[y; p]$ is supposed to be real single valued for all arguments (y, p) such that y is in $(y_0)_a$ and p in $\mathfrak{P}$, and to have the usual property that it is in the class $\mathfrak{M}$ when the argument y is fixed. It has furthermore a difference function A of the kind described above such that

$$G[y_1; p] - G[y_2; p] = A[y_1, y_2, y_1 - y_2; p]$$

whenever (y_1, p) and (y_2, p) are elements in the domain of definition of G. The functional $A[y_0, y_0, \eta; p]$ is called the differential of G at y_0. Since y_0 is a fixed element of the class $\mathfrak{M}$ the differential is a function of η and p alone.

The use which Goursat makes of his hypothesis concerning the non-vanishing of the functional determinant suggests the assumption that A has a "reciprocal" for $y_1 = y_2 = y_0$, namely that there exists a functional $\bar{A}[\eta; p]$ such that

$$\bar{A}[A[y_0, y_0, \eta; q]; p] = \eta(p)$$

$\bar{A}[\eta; p]$ has the properties (1) and (2) prescribed for A, where $\bar{M}$ denotes the number corresponding to the M of property (2). It has the further property that it vanishes identically in p only when $\eta(p) = 0$ for every p.

LEMMA 6. *The functional F defined by the equation*

$$F[y, z; p] = y - \bar{A}[G[y; p] - z; p]$$

has the properties of the functional F of § 2 near the element (y_0, z_0) where

$$z_0 = G[y_0; p].$$

As to the property (1) of § 2, it follows from the definition of F given in this lemma that

$$F[y_0, z_0; p] = y_0 - \bar{A}[0; p] = y_0.$$

The continuity, property 2, is proved by these inequalities,

$$\begin{aligned}
||F[y, z; p] - F[y', z'; p] \\
&= || y - y' + \bar{A}[G[y; p] - G[y'; p] - z + z'; p] || \\
&\leqq || y - y' || + \bar{M} || G[y; p] - G[y'; p] - z + z' || \\
&\leqq (1 + M\bar{M}) || y - y' || + || z - z' ||.
\end{aligned}$$

To find the K of property 3, use the linearity of the functional $\bar{A}$.

$$\begin{aligned}
|| F[y, z; p] - F[y', z; p] || &= || y - y' - \bar{A}[G[y; q] - G[y'; q]; p] || \\
&= || y - y' - \bar{A}[A[y, y', y - y'; q] - A[y_0, y_0, y - y'; q] \\
&\qquad + A[y_0, y_0, y - y'; q]; p] ||.
\end{aligned}$$

From linearity again, from the fact that $\bar{A}$ is the reciprocal of A, and from Postulate 3, $\|-y\| = \|y\|$, this expression reduces to

$$\|\bar{A}[A[y, y', y - y'; q] - A[y_0, y_0, y - y'; q]; p].$$

Because $\bar{A}$ is bounded, this is less than

$$\bar{M}\left\| A\left[y, y', \frac{y - y'}{\|y - y'\|}; p\right] - A\left[y_0, y_0, \frac{y - y'}{\|y - y'\|}; p\right]\right\| \|y - y'\|.$$

The number a of Lemma 1 is then chosen to make the coefficient of $\|y - y'\|$ less than $K < 1$.

THEOREM 3. *The solution of the equation*

$$y = F[y, z; p] \tag{5}$$

where F is defined in Lemma 6, *satisfies uniquely the equation*

$$G[y; p] = z(p), \tag{11}$$

and is continuous as a functional of z.

For, from the definition of F, (5) reduces to

$$\bar{A}[G[y; q] - z; p] = 0$$

and since $\bar{A}[\eta; p]$ vanishes identically only when $\eta(p) \equiv 0$, it follows that

$$G[y; p] \equiv z(p).$$

Any other function y', a solution of (11), would make F reduce to y', and would satisfy (5). But the solution of (5) is unique (Lemma 4). The solutions of (5) and (11) have been shown to be the same, and the solution of (5) is continuous (Lemma 5). This proves the continuity asserted in the theorem.

§ 4. *An Application to the Calculus of Variations.*

The theorem of § 3 will now be applied to the differential equations of the problem of Lagrange in the Calculus of Variations. For this problem the functions y in the integral

$$\int_a^b f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx,$$

to be minimized are subject to two sets of conditions. They must satisfy, first, the $m < n$ differential equations,

$$\varphi_\alpha(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \qquad (\alpha = 1, \cdots, m), \tag{12}$$

and second, the end conditions,

$$y_i(a) - h_i = 0, \tag{13}$$

$$y_i(b) - k_i = 0 \qquad (i = 1, \cdots, n). \tag{14}$$

The equation (12) may be regarded as a single equation in the composite variable (α, x), whose range is a subset of the range of elements (i, x) where $i = 1, 2, \cdots n, a \leqq x \leqq b$.

Bliss* has given a treatment of a problem of which this is a special case by adjoining to (12) the $n - m$ new equations

$$(15) \qquad \varphi_r(x, y_1, \cdots, y_n, y'_1, \cdots, y'_n) = Z_r(x) \quad (r = m + 1, \cdots, n).$$

In (15) the functions φ_r are arbitrary except that they are to be chosen so that the determinant $|\partial\varphi_i/\partial y'_j|$ is different from zero at every point of the minimizing arc to be studied. Equations (12) and (15) can then be written together in the single equation

$$(16) \qquad \varphi_i(x, y_1, \cdots, y_n, y'_1, \cdots, y'_n) = Z_i(x) \quad (i = 1, 2, \cdots, n),$$

with the understanding that $Z_i = 0$ identically in x, for $i \leqq m$.

Consider now a system of solutions $y_i^{(0)}(x)$, $Z_i^{(0)}(x)$ of class C' of the equations (16). In a neighborhood of the elements (x, y, y') of this solution the functions φ_i are supposed to have continuous first and second partial derivatives, and along the solution itself the functional determinant $|\partial\varphi_i/\partial y'_j|$ is different from zero. The partial derivatives $\partial\varphi_i/\partial y_j$ and $\partial\varphi_i/\partial y'_j$ will henceforth be denoted by φ_{ij} and ψ_{ij}, and their values at $x = a$, by $\varphi_{ij}(a)$ and $\psi_{ij}(a)$. It is proposed to show that the problem of determining a system of solutions of the equations (16) with initial conditions (13) is a special case of the theorem proved in § 3.

Equations (13) and (16) together are equivalent to the single system

$$G[y(q);\, p] = z(p),$$

where the independent variables are $p = (i, x)$, $q = (j, x_1)$ and G is the functional in the first member of the equation

$$(17) \quad \sum_j \psi_{ij}(a)(y_j(a) - h_j) + \int_a^x \varphi_i(x_1, y, y')dx_1 = z_i(x) \qquad (i = 1, \cdots, n).$$

Equations (13) have been multiplied by a matrix of rank n. The z_i appearing in (17) are the integrals from a to x of the functions $Z_i(x)$ in (16), and so vanish for $x = a$. Equations (14) are discussed later.

The general theory of the preceding sections will be applied to the solution of (17) for y when z is given. With the $y^{(0)}$ which minimizes the integral is associated a $z^{(0)}$ by equations (17), and it is in a first order neighborhood of these functions that a solution is to be found. The range $\mathfrak{P}$ is specified to be the set of elements (i, x), $(i = 1, \cdots, n;\ a \leqq x \leqq b)$. The class $\mathfrak{M}$ is the class of functions $y_i(x)$ which for each i are continuous with their first derivatives on the interval ab. The modulus, $||y||$, is

* *Transactions of the American Mathematical Society*, Vol. 19 (1918), p. 307.

the maximum of the absolute values of y_i and y'_i $(i = 1, \cdots, n)$. The functional $G[y; p]$ is the left-hand member of equations (17).

It remains to exhibit the differential A, its reciprocal $\bar{A}$, and to prove that the postulates of § 1 and hypotheses of § 2 are satisfied. Postulates 1–4 are immediately seen to be satisfied. Postulate 5 can be proven from the fact that convergence of the moduli of a sequence of functions of $\mathfrak{M}$ implies the uniform convergence of the functions and of their first derivatives.

The differential A is given for the function (17) by Taylor's formula* in the form

$$\sum_j \psi_{ij}(a)\eta_j(a) + \sum_j \int_a^x \{C_{ij}(x_1)\eta_j(x_1) + C'_{ij}(x_1)\eta'_j(x_1)\}dx_1, \tag{18}$$

where

$$C_{ij}(x_1) = \int_0^1 \varphi_{ij}(x_1, y^{(1)} + u(y^{(2)} - y^{(1)}), y^{(1)\prime} + u(y^{(2)\prime} - y^{(1)\prime}))du,$$

$$C'_{ij}(x_1) = \int_0^1 \psi_{ij}(x_1, y^{(1)} + u(y^{(2)} - y^{(1)}), y^{(1)\prime} + u(y^{(2)\prime} - y^{(1)\prime}))du.$$

In C and C', $y^{(1)}$ and $y^{(2)}$ are the arguments of the functional A, and are in a first order neighborhood of the extremal $y^{(0)}$ such that the determinant $|\psi_{ij}| \neq 0$, and φ is defined. When $y^{(1)} = y^{(2)} = y^{(0)}$, A reduces to

$$\sum_j \psi_{ij}(a)\eta_j(a) + \sum_j \int_a^x \{\varphi_{ij}\eta_j(x_1) + \psi_{ij}\eta'_j(x_1)\}dx_1. \tag{19}$$

To exhibit the reciprocal $\bar{A}$ is to define an operation which will reduce (19) to $\eta_k(x)$. This operation will be taken in the form

$$\bar{A} = \sum_i \left[l_{ki}(x)\eta_i(a) + \int_a^x \lambda_{ki}(x, x_1)\eta_i(x_1)dx_1 + \nu_{ki}(x)\eta_i(x) \right]$$

with suitably chosen functions l, λ, ν, and it is to be proved that when the functions $\eta(q) = \eta_i(x_1)$ of the variable $q \doteq (i, x_1)$ is replaced by A in this expression the result is $\eta(p)$ with $p = (k, x)$. To distinguish variables of integration from each other and from limits of integration, the notations x, x_1, x_2 are used. Summations are from 1 to n. To choose the functions l, λ, ν operate as follows. Put $x = a$ in (18) and multiply by undetermined factors $l_{ki}(x)$. Form (18) for x_1, $(a < x_1 < x)$, multiply by $\lambda_{ki}(x, x_1)$, and integrate from a to x. For $x_1 = x$, multiply by $\nu_{ki}(x)$. Add the terms so formed and sum as to i.

A method of choosing the functions l, λ and ν is to be given so that the expression,

$$\sum_{ij} \left[\left\{ l_{ki}(x) + \int_a^x \lambda_{ki}(x, x_1)dx_1 + \nu_{ki}(x) \right\} \psi_{ij}(a)n_i(a) \right.$$

* Jordan, *Cours d'Analyse*, 2d ed., Vol. 1, p. 247.

$$+\int_a^x \lambda_{ki}(x, x_1)\int_a^{x_1}(\varphi_{ij}\eta_j+\psi_{ij}\eta_j')_{x_2}dx_2dx_1$$
$$+\int_a^x \nu_{ki}(x)\{\varphi_{ij}\eta_j+\psi_{ij}\eta_j'\}_{x_2}dx_2\Big],$$

whose formation was described in the preceding paragraph, reduces to $\eta_k(x)$. By the change in order of integration in the second term, and the combination of the last two terms, this becomes

$$(20)\quad \sum_{ij}\Big[\Big\{l_{ki}(x)+\int_a^x \lambda_{ki}(x, x_1)dx_1+\nu_{ki}(x)\Big\}\psi_{ij}^{(a)}\eta_j(a)$$
$$+\int_a^x(\varphi_{ij}\eta_j+\psi_{ij}\eta_j')_{x_2}\Big\{\int_{x_2}^x \lambda_{ki}(x, x_1)dx_1+\nu_{ki}(x)\Big\}dx_2\Big].$$

A set of auxiliary functions $\mu_k(x, x_2)$ may be defined by means of the equations

$$(21)\qquad \int_{x_2}^x \lambda_{ki}(x, x_1)dx_1+\nu_{ki}(x)=\mu_{ki}(x, x_2)\quad (\kappa, i=1, 2, \cdots, n).$$

From (21) and the integration of the last term by parts, (20) is seen to become

$$(22)\quad \sum_{ij}\Big[\Big\{l_{ki}(x)+\int_a^x \lambda_{ki}(x, x_1)dx_1+\nu_{ki}(x)\Big\}\psi_{ij}(a)n_j(a)$$
$$+\int_a^x\Big\{\mu_{ki}(x, x_2)\psi_{ij}(x_2)-\int_a^{x_2}\mu_{ki}(x, x_1)\varphi_{ij}(x_1)dx_1\Big\}\eta_j'dx_2$$
$$+\eta_j(x)\int_a^x \mu_{ki}(x, x_1)\varphi_{ij}(x)dx_1\Big].$$

Next it will be shown that the functions $\mu_{ki}(x, x_1)$ can be so chosen that the brace under the integral in the second term is independent of x_2 and therefore equal to a function $\kappa_{ki}(x)$ satisfying the following equation:

$$(23)\qquad \sum_i \mu_{ki}(x, x_2)\psi_{ij}(x_2)=\sum_i\int_a^{x_2}\mu_{ki}(x, x_1)\psi_{ij}(x_1)dx_1+\kappa_{kj}(x),$$
$$(j=1, \cdots, n).$$

The differentiation of (23) for x_2 as it stands would imply the existence of y''. To avoid this replace the μ's by linear combinations of them, $v_{kj}(x, x_2)$, determined by the following equations,

$$(24)\qquad v_{kj}(x, x_2)=\sum_i \mu_{ki}(x, x_2)\psi_{ij}(x_2).$$

The solution of these for the functions μ is possible since $|\psi_{ij}|\neq 0$, and it gives

$$(25)\qquad \mu_{ki}(x, x_2)=\sum_r c_{kr}(x_2)v_{ri}(x, x_2).$$

From (24) and (25), equation (23) becomes

$$v_{kj}(x, x_2) = \sum_{ir} \int_a^{x_2} c_{kr}(x_1) v_{ri}(x, x_1) \varphi_{ij}(x_1) dx_1 + \kappa_{kj}(x).$$

In this equation the right member is differentiable for x_2, and the equations for the determination of the functions v_{kj} may be written in the form

$$\frac{d}{dx_2} v_{kj}(x, x_2) = \sum_{ir} c_{kr}(x_2) v_{ri}(x, x_2) \varphi_{ij}(x_2). \tag{26}$$

These are linear differential equations which determine $v_{kj}(x, x_2)$ uniquely subject to the initial conditions,

$$v_{kj}(x, x) = \delta_{kj}, \tag{27}$$

where δ_{kj} is unity when $\kappa = j$ and zero otherwise. When the functions v_{kj} are known the μ's are given by (25), the κ's by (23) and the λ's and v's by (21). With the help of (23), (24) and (27) the expression (22) may be replaced by

$$\sum_{ij} \left[\left\{ l_{ki}(x) + \nu_{ki}(x) + \int_a^x \lambda_{ki}(x, x_1) dx_1 \right\} \psi_{ij}(a) \eta_j(a) \right] - \sum_j \kappa_{kj}(x) \eta_j(a) + \eta_k(x). \tag{28}$$

The functions l may now be determined by the equation

$$\sum_i l_{ki}(x) \psi_{ij}(a) = \kappa_{kj}(x) - \sum_i \nu_{ki}(x) \psi_{ij}(a) - \sum_i \psi_{ij}(a) \int_a^x \lambda_{ki}(x, x_1) dx_1 \tag{29}$$

so that everything in the expression (28) disappears except $\eta_k(x)$. This result is formulated in the following definition and theorem.

Definition. The differential $A[y_0, y_0, \eta; p]$ of the functional $G[y; q]$ in the equation (17) for the problem of Lagrange is the expression

$$\sum_j \psi_{ij}(a) \eta_j(a) + \sum_j \int_a^x (\varphi_{ij} \eta_j + \psi_{ij} \eta_j')_{x_1} dx_1. \tag{30}$$

The functional $\bar{A}[\eta; p]$ is given by the formula

$$\sum_i \left[l_{ki}(x) n_i(a) + \int_a^x \lambda_{ki}(x, x_1) \eta_i(x_1) dx_1 + \nu_{ki}(x) \eta_i(x) \right]. \tag{31}$$

In this definition the functions φ_{ij} and ψ_{ij} are formed for the extremal $y^{(0)}$, the functions λ and ν are determined by the equations (26), (27), (25) and (21), and the functions l by (29).

THEOREM 4. *The functional $\bar{A}$ is the reciprocal of A, that is if the η in* (31) *is replaced by the function* (30), *then* (31) *will reduce to $\eta_k(x)$.*

The differential A given by (30) is seen to satisfy the first and second assumptions of § 3. The reciprocal $\bar{A}$ is also seen to satisfy these assump-

tions. The third assumption as to A follows from the continuity properties of φ, and from the mean value theorem. It remains to show that the reciprocal vanishes identically only with the argument η.

LEMMA 7. *If the functions $\eta_i(x)$ are continuous with their first derivatives on the interval ab, and if the equation*

$$(31)\qquad \sum_i \left[l_{ki}(x)\eta_i(a) + \int_a^x \lambda_{ki}(x, x_1)\eta_i(x_1)dx_1 + \nu_{ki}(x)\eta_i(x) \right] = 0$$

holds identically in κ and x, it follows that $\eta_i(x) = 0$ identically in i and x.

To prove this, put $x = a$. From (29) with the help of equations (21) and (23) for $x = x_2 = a$, it follows that $l_{ki}(a) = 0$, and from (24) and (27) it is seen that $|\nu_{ki}(a)| \neq 0$. Therefore $\eta_k(a) = 0$ identically in κ, and it is correct to write

$$\eta_i(x_1) = \int_a^{x_1} \eta_i'(x_2)dx_2.$$

From (31) then

$$\sum_i \int_a^x \lambda_{ki}(x, x_1) \int_a^{x_1} \eta_i'(x_2)dx_2dx_1 + \nu_{ki}(x) \int_a^x \eta_i'(x_2)dx_2 = 0.$$

By change of order of integration, combination of terms and the use of (21), this becomes

$$(32)\qquad \sum_i \int_a^x \mu_{ki}(x, x_1)\eta_i'(x_1)dx_1 = 0.$$

From the theory of differential equations, the solutions of equations (26), and hence also the functions $\mu_k(x, x_1)$, are differentiable for x. Then differentiation of (32) with respect to x gives

$$(33)\qquad \sum_i \mu_{ki}(x, x)\eta_i'(x) = -\sum_i \int_a^x \frac{\partial}{\partial x}\mu_{ki}(x, x_1)\eta_i'(x_1)dx_1.$$

After multiplying by $\bar{\mu}_{rk}(x)$, the matrix reciprocal to $\mu_{ki}(x, x)$, summing with respect to κ and setting

$$-\sum_k \bar{\mu}_{rk}(x)\frac{\partial}{\partial x}\mu_{ki}(x, x_1) = \sigma_{ri}(x, x_1)$$

the equations (33) give

$$(34)\qquad \eta_r'(x) = \sum_i \int_a^x \sigma_{ri}(x, x_1)\eta_i'(x_1)dx_1.$$

The proof that no solution of (32) exists except $\eta_r'(x)$ identically zero is a slight modification of the corresponding proof for Volterra's integral equation.* If M and m are the maxima of $|\sigma_{ri}(x, x_1)|$ and $n_i'(x)$ respectively, for $r, i = 1, 2 \cdots n$ and values of x and x_1 on the interval ab, the

* Bôcher, *An Introduction to the Study of Integral Equations*, p. 15.

equations (34) give

$$m \leqq \int_a^x nMmdx = nMm(x-a),$$

and by repeated applications of this inequality it follows that

$$m \leqq n^\alpha M^\alpha m \frac{(x-a)^\alpha}{\alpha!}$$

for every positive integer α. As this last expression approaches zero with increasing α, it follows that

$$\eta_r'(x) = 0, \qquad a \leqq x \leqq b, \qquad r = 1, \cdots, n.$$

Since $\eta_r(a) = 0$, it is true that $\eta_r(x) \equiv 0$, as stated in the lemma.

The postulates and hypotheses of the general theory have been proved to be satisfied in the case of the Lagrange problem. The results of this section may be stated in the following theorem.

THEOREM. *Under the hypotheses made at the beginning of this section the system of equations*

$$\varphi_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = Z_i(x) \qquad (i = 1, \cdots, n)$$

with the initial conditions $y_i(a) = h_i$, $(i = 1, \cdots, n)$, *is equivalent to the single equation*

$$\sum_j \psi_{ij}(a)[y_j(a) - h_j] + \int_a^x \varphi_i dx = z_i(x).$$

This has the form

$$G[y(q);\, p] = z(p)$$

where p and q represent the pairs $p = (i, x)$, $q = (j, x)$. If $y^{(0)}(q)$, $z^{(0)}(p)$ is an initial solution of the last equation with properties as prescribed above, then there exist two neighborhoods $(y^{(0)})_a$ and $(z^{(0)})_{a_1}$ such that to every $z(p)$ in the latter there corresponds one and but one solution $y(q)$ in $(y^{(0)})_a$. The functional $y(q) = Y[z; q]$ so defined is continuous in $(z^{(0)})_{a_1}$ and reduces to $y = y^{(0)}$ for $z = z^{(0)}$.

THE UNIVERSITY OF CHICAGO.

ON THE LAPLACE-POISSON MIXED EQUATION.

BY R. F. BORDEN.

INTRODUCTION.

We designate as the Laplace-Poisson mixed equation, the equation*

$$(1) \qquad f'(x+1) + p(x)f'(x) + q(x)f(x+1) + m(x)f(x) = 0.$$

which was first studied by Poisson†, and which is analogous in form and in theory to the Laplace partial differential equation

$$s + ap + bq + cz = 0.$$

Poisson finds solutions in finite form by means of transformations analogous to those used by Laplace in solving the differential equation written above. These transformations put the mixed equation into equations of the same form, viz:

$$F'(x+1) + P(x)F'(x) + Q(x)F(x+1) + M(x)F(x) = 0.$$

When certain relations exist between the coefficients of one of the transformed equations, Poisson solves that equation by the standard methods of solving linear difference and differential equations of the first order, and then obtains the solution of the original equation by reversing the transformations.

The remark of Poisson, that the theory of this type of equation is but little advanced, still holds true more than a century later. In this paper

* The coefficients $p(x)$, $q(x)$, and $m(x)$ are analytic functions of the real or complex variable x.

† *Jour. de l'École Polytechnique*, t. 6 (1806), pp. 127–141. See also Lacroix, "Traité du Calcul," 3d ed., Vol. 3, pp. 575–600, for the work of Poisson and other early investigators in the field. Other papers on mixed equations are the following: Vernier, *Ann. de Math.*, 13 (1882), 258–267; Gregory, *Cambridge Math. Jour.*, 1 (1839), 54; Boole, "A Treatise on the Calculus of Finite Differences" (1860); Walton, *Quart. Jour.*, *10* (1870), 248–253; Combescure, *Ann. Ec. Nor. Sup.* (2), 3, (1874), 305–362; Cesàro, *Nouv. Ann.* (3), 4, (1885), 36–41; Laurent, "Traité de Analyse" (1890), Vol. 6, 234–236; Lemeray, *Edinburgh Math. Soc. Proc.* (1898), 13–14; Lecornu, *Bull. de Soc. Math. de France*, 27, (1899), 153–160; Oltramare, *Assoc. Fr. Marseille*, 20 (1891), 66–82; Oltramare, *Bordeaux Assoc. Fr. Bull.*, 24 (1899), 175–186; Oltramare, "Calcul de Generalisation" (1899); Brajtzew, Moscow Coll. (1901); Pincherle, *Rendiconti Pal.*, 18 (1904); Pincherle, *Mem. Soc. Italiana d. Sc.* (3), 15 (1907); Meissner, *Schweiz. Bauzeitung.*, 54; Polussuchin, Zurich Diss. (1910); Schmidt, *Math. Ann.*, 70 (1911), 499–524; Bateman, Proc. 5th Int. Cong. Math., Vol. 1 (1912), 291–294; Haag, *Bull. de Soc. Math.*, 36 (1912), 10–24; Schurer, *Ber. Gew. Wiss. Leipzig* (1912), 167–236; (1913), 139–143; (1914), 137–158; Carmichael, *Am. Jour.*, 35 (1913), 151–162; Bennett, *Ann. of Math.* (2), 18.

the elementary theory of the equation is extended along lines initiated by Poisson. Most of Poisson's results are incidentally included, but the work is from a different point of view, and the formulas obtained are more explicit, since their explicit forms are needed in the development of our further results. The theory of the invariants under the group of transformations $f(x) = v(x)g(x)$ is developed along the same lines as is the corresponding theory of the Laplace equation.* Largely the same methods are used, the analogy being very close. The results are summarized below by sections.

I. The functions

$$\frac{m(x)}{p(x)} - q(x) - \frac{p'(x)}{p(x)}, \quad \text{and} \quad \frac{m(x)}{p(x)} - q(x-1)$$

form a fundamental set of invariants under the group of transformations $f(x) = v(x)g(x)$, which transformations do not change the form of the equation.

II. When one of the fundamental invariants is zero the equation is of such nature that it may be obtained by differentiating a difference equation or else by applying the difference operation to a differential equation. That is, it may be solved by integrating first a linear differential [difference] equation and then a linear difference [differential] equation. These solutions each involve an arbitrary constant and an arbitrary periodic function of period one.

III. The Laplace-Poisson transformations

$$(S)\ f_{S_1}(x) = f(x+1) + p(x)f(x) \quad \text{and} \quad (T)\ f_{T_1}(x) = f'(x) + q(x-1)f(x)$$

leave the form of the equation unchanged. The invariants of the equation gotten by applying S or T are simply expressible in terms of the invariants of the original equation. The two transformations are, in a sense, inverses of each other; for the application of both in either order to (1) gives an equation with the same invariants as (1). Successive applications of S, or of T, give equations of the same type, whose invariants can be expressed in terms of the invariants of the preceding equations under the successive transformations, and therefore in terms of the invariants of the original equation.

IV. The solutions of the equations obtained by successive applications of S or T may be obtained in terms of the solution of the nth transformed equation. In particular, the solution of the original equation may be thus obtained.

V. The term *rank* of the equation is introduced in accordance with the nomenclature of the corresponding theory of the Laplace partial differential

* An exposition of the theory of the partial differential equation $s + ap + bq + cz = 0$ may be found in Forsyth's "Theory of Differential Equations," Vol. 6, pp. 44–96.

equation.* The mixed equation is said to be of finite rank when a finite number of applications of S, or of T, results in an equation with a vanishing invariant. The equation can then be solved in finite form, and an arbitrary part of the solution can be so chosen that quadratures of arbitrary functions are not involved. The equation is said to be of rank $n+1$ of the first kind when n applications of S give an equation with a vanishing invariant. This is a necessary and sufficient condition for a solution of the form

$$f(x) = E_0(x)F(x) + E_1(x)F'(x) + \cdots + E_n(x)F^{(n)}(x),$$

where the E's are determinate functions and $F(x)$ is an arbitrary periodic function of period one. The mixed equation is said to be of rank $n+1$ of the second kind when n applications of T give an equation with a vanishing invariant. In this case, the solution without quadrature of arbitrary functions is a determinate function of x multiplied by an arbitrary constant.

VI, VII. The restrictions on the coefficients of the mixed equation in order that it be of finite rank of the first kind or of the second kind are found.

VIII. When the equation is of finite rank with respect to both S and T, it is said to be of doubly finite rank. The restrictions on the coefficients of such an equation are found.

IX. Generalizations of the Laplace-Poisson transformations analogous to the transformations used by Lévy† in connection with the Laplace differential equation are here tried with a result similar to that found by Lévy, viz: that they are not generally useful in obtaining an equation of the type (1) with a vanishing invariant.‡

I. The Invariants of the Equation.

The equation§

$$f'(x+1) + p(x)f'(x) + q(x)f(x+1) + m(x)f(x) = 0 \tag{1}$$

is put by a transformation of the group

$$f(x) = v(x)g(x)$$

into the form

$$g'(x+1) + P(x)g'(x) + Q(x)g(x+1) + M(x)g(x) = 0, \tag{2}$$

* Forsyth, l. c., p. 60.

† *Journal de l'École Polytechnique*, t. 38 (1886), p. 67.

‡ See Forsyth, l.c., p. 94.

§ We shall develop the theory only for the case when $p(x)$ is not zero. When $p(x) = 0$, $I(x)$ and $J(x)$ are illusory and $\alpha(x)$ and $\beta(x)$ are each equal to $m(x)$. It may be readily seen by following through this paper that the case $p(x) = 0$ can be carried, but the conditions for solutions here developed become simply $m(x) = 0$ when $p(x) = 0$, and the equation is then not a true mixed equation.

where

$$P(x) = p(x)\frac{v(x)}{v(x+1)}, \qquad Q(x) = \frac{v'(x+1)}{v(x+1)} + q(x),$$

and

$$M(x) = m(x)\frac{v(x)}{v(x+1)} + p(x)\frac{v'(x)}{v(x+1)}.$$

The form of the equation is therefore unaltered by the substitution. By eliminating $v(x)$ in two ways from the relations between the coefficients of (1) and (2), we may obtain

$$p(x)[M(x) - P(x)Q(x) - P'(x)] = P(x)[m(x) - p(x)q(x) - p'(x)].$$

and

$$p(x)[M(x) - P(x)Q(x-1)] = P(x)[m(x) - p(x)q(x)1)].$$

Hence

$$I(x) = \frac{m(x)}{p(x)} - q(x) - \frac{p'(x)}{p(x)}$$

and

$$J(x) = \frac{m(x)}{p(x)} - q(x-1)$$

are absolute invariants of the equation (1) under the group of transformations $f(x) = v(x)g(x)$. We shall find it convenient to use also the relative invariants

$$\alpha(x) = m(x) - p(x)q(x) - p'(x) \qquad \text{and} \qquad \beta(x) = m(x) - p(x)q(x-1).$$

These functions are each multiplied by $v(x)/v(x+1)$ at each application of $f(x) = v(x)g(x)$.

We will now show that $I(x)$ and $J(x)$ form a fundamental set of invariants of the equation (1); i.e., that all invariants of (1) under $f(x) = v(x)g(x)$ can be expressed as functions of $I(x)$ and $J(x)$ involving only algebraic operations, the operations of the differential calculus and the difference calculus and their inverses.

We will choose $v(x)$ in the transformation $f(x) = v(x)g(x)$ so that the equation (1) will be put into the form (2) subject to the restriction

$$P(x)Q(x) = M(x).$$

This condition reduces to

$$\frac{v(x)}{v(x+1)}[p(x)q(x) - m(x)] = p(x)\frac{d}{dx}\left[\frac{v(x)}{v(x+1)}\right],$$

whence we may take

$$\frac{v(x)}{v(x+1)} = e^{\int_{x_0}^{x}\frac{p(x)q(x) - m(x)}{p(x)}dx}.$$

This condition is also sufficient. So, by choosing $v(x)$ properly, we can transform the equation (1) into the form

$$g'(x+1) + P(x)g'(x) + Q(x)g(x+1) + P(x)Q(x)g(x) = 0.$$

The I and J invariants of this equation are

$$I(x) = -\frac{P'(x)}{P(x)}$$

and

$$J(x) = Q(x) - Q(x-1) = \Delta Q(x-1).$$

Accordingly

$$P(x) = ce^{-\int_{x_0}^{x} I(x)dx},$$

and

$$Q(x) = \Sigma J(x+1),$$

where Σ denotes some particular finite integral. Hence the transformed equation is

$$(3) \quad g'(x+1) + e^{-\int_{x_0}^{x} I(x)dx} g'(x) + \Sigma J(x+1)g(x+1) + e^{-\int_{x_0}^{x} I(x)dx} \Sigma J(x+1)g(x) = 0.$$

This is of the same form as the original equation (1), and is derived from (1) by a transformation of the group $f(x) = v(x)g(x)$. Therefore (3) has the same invariants as (1) under transformations of the type considered. Since the invariants are functions of the coefficients alone, it follows that all the invariants of (3) are expressible in terms of $I(x)$ and $J(x)$ only. We shall refer to $I(x)$ and $J(x)$ simply as the invariants of the equation.

II. Solutions when One Invariant is Zero.

If $I(x) = 0$, then $\alpha(x) = 0$,

and

$$m(x) = p'(x) + p(x)q(x).$$

The equation may then be written in the form

$$\frac{d}{dx}[f(x+1) + p(x)f(x)] + q(x)[f(x+1) + p(x)f(x)] = 0,$$

whence

$$f(x+1) + p(x)f(x) = ce^{-\int_{x_0}^{x} q(x)dx}.$$

To solve this, we first solve the homogeneous equation

$$g(x+1) - [-p(x)]g(x) = 0$$

as follows

$$\log g(x+1) - \log g(x) = \log[-p(x)]$$

or

$$g(x) = \varphi(x)e^{\Sigma \log[-p(x)]},$$

where $\varphi(x)$ is an arbitrary periodic function of period one, and Σ denotes a finite integral* for some range of the variable x.

Let $f(x) = u(x)g(x)$ and substitute in the non-homogeneous equation. We get, taking $\varphi(x) = 1$,

$$u(x+1)e^{\Sigma \log[-p(x+1)]} - [-p(x)]u(x)e^{\Sigma \log[-p(x)]} = ce^{-\int_{x_0}^{x} q(x)dx}.$$

Hence

$$u(x+1) - u(x) = ce^{-\int_{x_0}^{x} q(x)dx - \Sigma \log[-p(x+1)]},$$

and therefore

$$u(x) = \Sigma\left[ce^{-\int_{x_0}^{x} q(x)dx - \Sigma \log[-p(x+1)]}\right] + F(x)$$

where $F(x)$ has the period one and is otherwise arbitrary. So we have

$$(4) \qquad f(x) = e^{\Sigma \log[-p(x)]}\left\{F(x) + \Sigma ce^{-\int_{x_0}^{x} q(x)dx - \Sigma \log[-p(x+1)]}\right\}.$$

If $J(x) = 0$, then $\beta(x) = 0$, and

$$m(x) = p(x)q(x-1).$$

The equation may then be written

$$f'(x+1) + q(x)f(x+1) + p(x)[f'(x) + q(x-1)f(x)] = 0,$$

from which we obtain†

$$\Delta \log[f'(x) + q(x-1)f(x)] = \log[-p(x)],$$

whence

$$f'(x) + q(x-1)f(x) = \theta(x)e^{\Sigma \log[-p(x)]},$$

$\theta(x)$ being an arbitrary periodic function of period one. Hence we have

$$(5) \quad f(x) = e^{-\int_{x_0}^{x} q(x-1)dx}\int_{x_0}^{x} \theta(x)e^{\Sigma \log[-p(x)] + \int_{x_0}^{x} q(x-1)dx} + ke^{-\int_{x_0}^{x} q(x-1)dx},$$

where K is an arbitrary constant.

* $F(x)$ is said to be a finite integral of $G(x)$ if

$$F(x+1) - F(x) = G(x).$$

In this paper, the symbol Σ without limits of summation denotes a finite integral. When used with limits, e.g., $\sum_{k=0}^{n}$, it denotes an ordinary summation.

† Δ denotes the difference of a function, i.e.,

$$\Delta v(x) = v(x+1) - v(x).$$

We have thus shown that when $I(x) = 0$ [$J(x) = 0$] a solution of the equation (1) can be obtained in finite form by solving, first a linear differential [difference] equation of the first order, and then a linear difference [differential] equation of the first order.

III. The Laplace-Poisson Transformations and the Invariants of the Resulting Equation.

The Laplace-Poisson transformation

$$F_{S_1}(x) = f(x+1) + p(x)f(x) \tag{S}$$

transforms the equation (1) into an equation of the same form, viz:

$$f'_{S_1}(x+1) + p_{S_1}(x)f'_{S_1}(x) + q_{S_1}(x)f_{S_1}(x+1) + m_{S_1}(x)f_{S_1}(x) = 0, \tag{6}$$

where

$$p_{S_1}(x) = p(x)\frac{\alpha(x+1)}{\alpha(x)} = p(x+1)\frac{I(x+1)}{I(x)},$$
$$q_{S_1}(x) = q(x+1),$$

and

$$m_{S_1}(x) = p(x+1)q(x)\frac{I(x+1)}{I(x)} + p(x+1)I(x+1).$$

The invariants of (6) under the group $f_{S_1}(x) = v(x)g(x)$ are

$$\begin{aligned} J_{S_1}(x) &= \frac{m_{S_1}(x)}{p_{S_1}(x)} - q_{S_1}(x-1) \\ &= q(x) + I(x) - q(x) \\ &= I(x). \end{aligned}$$

and

$$\begin{aligned} I(x) &= \frac{m_{S_1}(x)}{p_{S_1}(x)} - q_{S_1}(x) - \frac{p'_{S_1}(x)}{p_{S_1}(x)} \\ &= q(x) + I(x) - q(x+1) - \frac{p'(x+1)}{p(x+1)} - \Delta\frac{d}{dx}\log I(x). \end{aligned}$$

If we add and subtract $m(x+1)/p(x+1)$, we get

$$I_{S_1}(x) = I(x+1) + I(x) - J(x+1) - \Delta\frac{d}{dx}\log I(x).$$

Under the Laplace-Poisson transformation

$$f_{T_1}(x) = f'(x) + q(x-1)f(x) \tag{T}$$

the equation (1) becomes

$$f'_{T_1}(x+1) + p_{T_1}(x)f'_{T_1}(x) + q_{T_1}(x)f_{T_1}(x+1) + m_{T_1}(x)f_{T_1}(x) = 0, \tag{7}$$

in which the coefficients may be reduced to the following forms:

$$p_{T_1}(x) = p(x),$$
$$q_{T_1}(x) = q(x-1) - \frac{J'(x)}{J(x)} - \frac{p'(x)}{p(x)},$$

and

$$m_{T_1}(x) = p(x)J(x) + p(x)q(x-1) - p(x)\frac{J'(x)}{J(x)}.$$

The invariants of (7) under the group $f(x)\ v(x)g(x)$ are

$$\begin{aligned} I_{T_1}(x) &= \frac{m_{T_1}(x)}{p_{T_1}(x)} - q_{T_1}(x) - \frac{p'_{T_1}(x)}{p_{T_1}(x)} \\ &= J(x) + q(x-1) - \frac{J'(x)}{J(x)} - q(x-1) + \frac{p'(x)}{p(x)} + \frac{J'(x)}{J(x)} - \frac{p'(x)}{p(x)} \\ &= J(x) \end{aligned}$$

and

$$\begin{aligned} J_{T_1}(x) &= \frac{m_{T_1}(x)}{p_{T_1}(x)} - q_{T_1}(x-1) \\ &= J(x) + q(x-1) - \frac{J'(x)}{J(x)} + \frac{J'(x-1)}{J(x-1)} + \frac{p'(x-1)}{p(x-1)} - q(x-2). \end{aligned}$$

If we add and subtract $m(x-1)/p(x-1)$, we get

$$J_{T_1}(x) = J(x) + J(x-1) - I(x-1) - \Delta\frac{d}{dx}\log J(x-1).$$

Hence we see that the transformations S and T each transform the equation (1) into an equation of the same form, the invariants of which may be simply expressed in terms of the invariants of (1).

The two transformations S and T are, in a sense, inverses of each other; for TS gives

$$\begin{aligned} f_{TS}(x) &= f'_{S_1}(x) + q(x)f_{S_1}(x) \\ &= f'(x+1) + p(x)f'(x) + p'(x)f(x) + q(x)f(x+1) + p(x)q(x)f(x) \\ &= [p'(x) + p(x)q(x) - m(x)]f(x) \\ &= -\,p(x)I(x)f(x) = -\,\alpha(x)f(x), \end{aligned}$$

and ST gives

$$\begin{aligned} f_{ST}(x) &= f_{T_1}(x+1) + p(x)f_{T_1}(x) \\ &= f'(x+1) + q(x)f(x+1) + p(x)f'(x) + p(x)q(x-1)f(x) \\ &= [p(x)q(x-1) - m(x)]f(x) \\ &= -\,p(x)J(x)f(x) = -\,\beta(x)f(x). \end{aligned}$$

Hence the equations resulting from applications of TS and ST have the same invariants as has the original equation. Furthermore we may transform the equation (1) into itself as follows. Apply $TS[ST]$. The resulting equation is that obtained by replacing $f(x)$ in (1) by $\alpha(x)f(x)[\beta(x)f(x)]$. Then the transformation $f(x) = g(x)/\alpha(x)[f(x) = g(x)/\beta(x)]$ brings us back to the equation (1).

Let $I_{S_n}(x)$ and $J_{S_n}(x)$ be the invariants of the equation obtained by n successive applications of S. Then we have

$$J_{S_n}(x) = I_{S_{n-1}}(x),$$

and

$$I_{S_n}(x) = I_{S_{n-1}}(x+1) + I_{S_{n-1}}(x) - J_{S_{n-1}}(x) - \Delta \frac{d}{dx} \log I_{S_{n-1}}(x).$$

So we can write

$$I_{S_n}(x) - I_{S_{n-1}}(x+1) - I_{S_{n-1}}(x) + I_{S_{n-2}}(x+1) = -\Delta \frac{d}{dx} \log I_{S_{n-1}}(x)$$

$$I_{S_{n-1}}(x) - I_{S_{n-2}}(x+1) - I_{S_{n-2}}(x) + I_{S_{n-3}}(x+1) = -\Delta \frac{d}{dx} \log I_{S_{n-2}}(x)$$

. . . .

. . . .

$$I_{S_1}(x) - I(x+1) - I(x) + J(x+1) = -\Delta \frac{d}{dx} \log I(x).$$

Adding these, we get

$$I_{S_n}(x) - I_{S_{n-1}}(x+1) = I(x) - J(x+1) - \Delta \frac{d}{dx} \log [I(x) I_{S_1}(x) \cdots I_{S_{n-1}}(x)].$$

Write this for $n, n-1, n-2, \cdots$ successively, and add 1 to the argument at each step. We then have

$$I_{S_n}(x) - I_{S_{n-1}}(x+1) = I(x) - J(x+1) - \Delta \frac{d}{dx} \log [I(x) I_{S_1}(x) \cdots I_{S_{n-1}}(x)]$$

$$I_{S_{n-1}}(x+1) - I_{S_{n-2}}(x+2)$$
$$= I(x+1) - J(x+2) - \Delta \frac{d}{dx} \log [I(x+1) \cdots I_{S_{n-2}}(x+1)]$$

. . . .

. . . .

$$I_{S_1}(x+n-1) - I(x+n)$$
$$= I(x+n-1) - J(x+n) - \Delta \frac{d}{dx} \log I(x+n-1).$$

Adding, we get

$$I_{S_n}(x) - I(x+n) = \sum_{k=0}^{n-1} I(x+k) - \sum_{k=1}^{n} J(x+k)$$
$$- \Delta \frac{d}{dx} \log \left[\prod_{k=0}^{n-1} I(x+k) \prod_{k=0}^{n-2} I_{S_1}(x+k) \cdots I_{S_{n-1}}(x) \right],$$

or

$$I_{S_n}(x) = I(x) + \sum_{k=1}^{n} [I(x+k) - J(x+k)] \tag{8}$$
$$- \Delta \frac{d}{dx} \log \left[\prod_{k=0}^{n-1} I(x+k) \prod_{k=0}^{n-2} I_{S_1}(x+k) \prod_{k=0}^{n-3} I(x+k) \cdots I_{S_{n-1}}(x)] \right].$$

Also

$$J_S(x) = I_{S_{n-1}}(x) = I(x) + \sum_{k=1}^{n-1} [I(x+k) - J(x+k)] \tag{9}$$
$$- \Delta \frac{d}{dx} \log \left[\prod_{k=0}^{n-2} I(x+k) \prod_{k=0}^{n-3} I_{S_1}(x+k) \cdots I_{S_{n-2}}(x) \right].$$

If $I_{T_n}(x)$ and $J_{T_n}(x)$ are the invariants of the equation obtained by applying T n times, we have

$$I_{T_n} = J_{T_{n-1}}(x),$$

and

$$J_{T_n}(x) = J_{T_{n-1}}(x) + J_{T_{n-1}}(x-1) - I_{T_{n-1}}(x-1) - \Delta \frac{d}{dx} \log J_{T_{n-1}}(x-1).$$

So we may write

$$J_{T_n}(x) - J_{T_{n-1}}(x-1) - J_{T_{n-1}}(x) + J_{T_{n-2}}(x-1) = -\Delta \frac{d}{dx} \log J_{T_{n-1}}(x-1)$$

$$J_{T_{n-1}}(x) - J_{T_{n-2}}(x-1) - J_{T_{n-2}}(x) + J_{T_{n-3}}(x-1) = -\Delta \frac{d}{dx} \log J_{T_{n-2}}(x-1)$$

. . . .

. . . .

$$J_{T_1}(x) - J(x-1) - J(x) + I(x-1) = -\Delta \frac{d}{dx} \log J(x-1).$$

Adding these, we get

$$J_{T_n}(x) - J_{T_{n-1}}(x-1) = J(x) - I(x-1)$$
$$- \Delta \frac{d}{dx} \log [J(x-1) J_{T_1}(x-1) \cdots J_{T_{n-1}}(x-1)].$$

Write this for n, $n-1$, $n-2$, $\cdots$ successively, subtracting 1 from the argument at each step. We have then

$$J_{T_n}(x) - J_{T_{n-1}}(x-1) = J(x) - I(x-1)$$
$$- \Delta \frac{d}{dx} \log [J(x-1) J_{T_1}(x-1) \cdots J_{T_{n-1}}(x-1)]$$

$$J_{T_{n-1}}(x-1) - J_{T_{n-2}}(x-2) = J(x-1) - I(x-2)$$
$$- \Delta \frac{d}{dx} \log [J(x-2) J_{T_1}(x-2) \cdots J_{T_{n-2}}(x-2)]$$

. . . .

. . . .

$$J_{T_1}(x-n+1) - J(x-n) = J(x-n+1) - I(x-n)$$
$$- \Delta \frac{d}{dx} \log [J(x-n)].$$

Adding these, we get

$$J_{T_n}(x) - J(x-n) = \sum_{k=0}^{n-1} J(x-k) - \sum_{k=1}^{n} I(x-k)$$
$$- \Delta \frac{d}{dx} \log \left[\prod_{k=1}^{n} J(x-k) \prod_{k=1}^{n-1} J_{T_1}(x-k) \cdots J_{T_{n-1}}(x-1) \right],$$

and therefore

$$(10) \quad J_{T_n}(x) = J(x) + \sum_{k=1}^{n} [J(x-k) - I(x-k)]$$
$$- \Delta \frac{d}{dx} \log \left[\prod_{k=1}^{n} J(x-k) \prod_{k=1}^{n-1} - J_{T_1}(x-k) \cdots J_{T_{n-1}}(x-1) \right].$$

Also

$$(11)\quad I_{T_n}(x) = J_{T_{n-1}}(x) = J(x) + \sum_{k=1}^{n-1}[J(x-k) - I(x-k)]$$
$$- \Delta\frac{d}{dx}\log\left[\prod_{k=1}^{n-1} J(x-k)\prod_{k=1}^{n-2} J_{T_2}(x-k)\cdots J_{T_{n-2}}(x-1)\right].$$

So we have the result that after n successive transformations of the equation (1) by $S[T]$, we arrive at an equation whose invariants can be expressed explicitly in terms of the invariants of (1) and the invariants of the intermediate equations obtained by 1, 2, 3, $\cdots$, $n-1$ applications of $S[T]$, and hence in terms of the invariants of the original equation.

IV. Solutions of Successively Transformed Equations.

After $n+1$ applications of S the new dependent variable is $f_{S_{n+1}}(x)$. Operate with T and call the resulting dependent variable $f_{S_{n+1},\,T_1}(x)$. We have then

$$f_{S_{n+1}}(x) = f_{S_n}(x+1) + p_{S_n}(x)f_{S_n}(x),$$
$$f_{S_{n+1},\,T_1}(x) = f'_{S_{n+1}}(x) + q_{S_{n+1}}(x-1)f_{S_{n+1}}(x)$$
$$= -\,p_{S_n}(x)I_{S_n}(x)f_{S_n}(x).$$

Multiplying by $\exp\left[\int_{x_0}^{x} q(x+n)dx\right]$, and remembering that

$$q_{S_{n+1}}(x-1) = q(x+n),$$

we have

$$\frac{d}{dx}\left[f_{S_{n+1}}(x)e^{\int_{x_0}^{x} q(x+n)dx}\right] = -\,p_{S_n}(x)I_{S_n}(x)f_{S_n}(x)e^{\int_{x_0}^{x} q(x+n)dx},$$

and therefore

$$f_{S_n}(x)e^{\int_{x_0}^{x} q(x+n)dx} = \frac{-1}{p_{S_n}(x)I_{S_n}(x)}\frac{d}{dx}\left[f_{S_{n+1}}(x)e^{\int_{x_0}^{x} q(x+n)dx}\right].$$

Since

$$e^{\int_{x_0}^{x} q(x+n)dx} = e^{\int_{x_0}^{x}[q(x+n)-q(x+n-1)+q(x+n-1)]dx}$$
$$= e^{\int_{x_0}^{x} q(x+n-1)dx}\, e^{\int_{x_0}^{x}\Delta q(x+n-1)dx}.$$

We may write

$$f_{S_n}(x)e^{\int_{x_0}^{x} q(x+n-1)dx} = A_n\frac{dB_n}{dx},$$

where

$$A_n = \frac{e^{\int_{x_0}^{x}\Delta q(x+n-1)dx}}{-\,p_{S_n}(x)I_{S_n}(x)}$$

and

$$B_n = f_{S_{n+1}}(x)e^{\int_{x_0}^{x} q(x+n)dx}.$$

Then we have

$$
(12)\quad
\begin{aligned}
f_{S_{n-1}}(x)e^{\int_{x_0}^{x} q(x+n-2)dx} &= A_{n-1}\frac{d}{dx}\left(A_n \frac{d}{dx} B_n\right),\\
f_{S_{n-2}}(x)e^{\int_{x_0}^{x} q(x+n-3)dx} &= A_n \frac{d}{dx}\left[A_{n-1}\frac{d}{dx}\left(A_n \frac{d}{dx} B_n\right)\right],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
f(x)e^{\int_{x_0}^{x} q(x-1)dx} &= A_0 \frac{d}{dx}\left\{A_1 \frac{d}{dx}\left[\cdots \frac{d}{dx}\left(A_n \frac{d}{dx} B_n\right)\right]\right\},
\end{aligned}
$$

where the successive A's are gotten by replacing n in A_n by $n-1$, $n-2$, $n-3, \cdots, 2, 1, 0$, if we agree that $p_{S_0}(x) = p(x)$ and $I_{S_0}(x) = I(x)$.

In a similar way we can get an expression for $f(x)$ in terms of $f_{T_n}(x)$, the dependent variable after n applications of T. The $(n+1)$th application of T gives

$$f_{T_{n+1}}(x) = f_{T_n}(x) + q_{T_n}(x-1)f_{T_n}(x).$$

Then operating with S, we get, since $p_{T_{n+1}}(x) = p(x)$,

$$
\begin{aligned}
f_{T_{n+1},\,S_1}(x) &= f_{T_{n+1}}(x+1) + p(x)f_{T_{n+1}}(x)\\
&= -\,p(x)J_{T_n}(x)f_{T_n}(x).
\end{aligned}
$$

We want to find a quantity $\chi(x)$ such that

$$\chi(x)[f_{T_{n+1}}(x+1) + p(x)f_{T_{n+1}}(x)] = \Delta[f_{T_{n+1}}(x)\psi(x)].$$

We may then take

$$\chi(x) = \psi(x+1),$$

and

$$p(x)V(x) = -\,\psi(x).$$

These two conditions give

$$\psi(x) = e^{-\Sigma \log[-p(x)]}.$$

Then we have

$$\chi(x) = \psi(x+1) = e^{-\Sigma \log[-p(x+1)]}.$$

So we write

$$[f_{T_{n+1}}(x+1) + p(x)f_{T_{n+1}}(x)]e^{-\Sigma \log[-p(x+1)]} = -\,p(x)J_{T_n}(x)e^{-\Sigma \log[-p(x+1)]}f_{T_n}(x),$$

whence

$$\Delta f_{T_{n+1}}(x)e^{-\Sigma \log[-p(x)]} = -\,p(x)J_{T_n}(x)f_{T_n}(x)e^{-\Sigma \log[-p(x+1)]},$$

and therefore

$$f_{T_n}(x)e^{-\Sigma \log[-p(x+1)]} = C_n \Delta D_n,$$

where

$$C_n = \frac{-1}{p(x)J_{T_n}(x)},$$

and

$$D_n = f_{T_{n+1}}(x)e^{-\Sigma \log[-p(x)]},$$

Then we have

$$\begin{aligned} f_{T_{n-1}}(x)e^{-\Sigma \log[-p(x+1)]} &= C_{n-1}\Delta(C_n\Delta D_n) \\ f_{T_{n-2}}(x)e^{-\Sigma \log[-p(x+1)]} &= C_{n-2}\Delta[C_{n-1}\Delta(C_n\Delta D_n)] \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

$$f(x)e^{-\Sigma \log[-p(x+1)]} = C_0\Delta\{C_1\Delta[\cdots \Delta(C_n\Delta D_n)]\}, \tag{13}$$

where the C's are gotten by replacing n in C_n by $n - 1, n - 2, \cdots, 2, 1, 0,$ if we agree that $J_{T_0}(x) = J(x)$.

Now we have expressed the solution $f(x)$ of the equation (1) in terms of $f_{S_n}(x)[f_{T_n}(x)]$, the solution of the nth transformed equation under $S[T]$. We have seen that we can find $f_{S_n}(x)[f_{T_n}(x)]$ if $I_{S_n}(x)=0[I_{T_n}(x)=0]$ or if $J_{S_n}(x) = 0$ $[J_{T_n}(x) = 0]$, and these solutions will be in finite form.

V. THE RANK OF THE EQUATION.

Suppose $I_{S_n}(x) = 0$. The equation may then be written in the form

$$\frac{d}{dx}[f_{S_n}(x+1) + p_{S_n}(x)f_{S_n}(x)] + q_{S_n}(x)[f_{S_n}(x+1) + p_{S_n}(x)f_{S_n}(x)] = 0,$$

which, as may be seen from § II, has a solution of the form

$$f_{S_n}(x) = e^{\Sigma \log[-p_{S_n}(x)]}\left\{F(x) + \Sigma c e^{-\int_{x_0}^{x} q_{S_n}(x)dx - \Sigma \log[-p_{S_n}(x+1)]}\right\},$$

where $F(x)$ has the period one and is otherwise arbitrary, and where c is an arbitrary constant.

As before, denote $e^{\int_{x_0}^{x} q(x+n-1)dx} f_{S_n}(x)$ by B_{n-1}. Then this expression is seen to be of the form

$$B_{n-1} = \eta(x)[F(x) + c\Sigma\xi(x)],$$

where $\eta(x)$ and $\xi(x)$ are determinate functions of x. Making use of a previously derived formula, viz:

$$e^{\int_{x_0}^{x} q(x-1)dx} f(x) = A_0\frac{d}{dx}\left\{A_1\frac{d}{dx}\left[\cdots\frac{d}{dx}\left(A_n\frac{d}{dx}B_n\right)\right]\right\}, \tag{12}$$

we get an expression for $f(x)$ of the form

$$\begin{aligned} f(x) = E_0(x)\{F(x) + c\Sigma\xi(x)\} &+ E_1(x)\{F'(x) + [c\Sigma\xi(x)]'\} \\ &+ E_2(x)[F''(x) + [c\Sigma\xi(x)]''\} + \cdots \\ &+ \cdots + \cdots \\ &+ \cdots + E_n(x)\{F^{(n)}(x) + [c\Sigma\xi(x)]^{(n)}\}, \end{aligned}$$

where the E's are determinate functions of x.

Taking the particular integral for which $c = 0$, we have a simpler integral which does not involve $\Sigma\xi(x)$, viz:

$$(14) \qquad f(x) = E_0(x)F(x) + E_1(x)F'(x) + \cdots + E_n(x)F^{(n)}(x).$$

We will call this an integral of *rank* $n + 1$ of the *first kind*. We will call the original equation of rank $n + 1$ of the first kind when $I_{S_n}(x) = 0$* and $I_{S_k}(x) \neq 0$, where k takes the values $0, 1, 2, \cdots, n - 1$.

Conversely, if the original equation

$$(1) \qquad f'(x + 1) + p(x)f'(x) + q(x)f(x + 1) + m(x)f(x) = 0$$

has a solution of the form (14), where the E's are determinate functions of x, and $F(x)$ is an arbitrary periodic function of period one; then after at most n applications of the transformation S, we will have an equation of which the I invariant is zero, i.e.,

$$I_{S_\mu}(x) = 0, \qquad (\mu \leqq n).$$

To show this, substitute in (1) the expression for $f(x)$ in (14), remembering that

$$F(x + \mu) = F(x).$$

We then have

$$\begin{aligned} E_0'(x+1)F(x)+E_0(x+1)F'(x)+ \cdots +E_n'(x+1)F^{(n)}(x)+E_n(x+1)F^{(n+1)}(x) \\ + p(x)[E_0'(x)F(x)+E_0(x)F'(x)+ \cdots E_n'(x)F^{(n)}(x)+E_n(x)F^{(n+1)}(x)] \\ + q(x)[E_0(x+1)F(x)+E_1(x+1)F'(x)+ \cdots +E_n(x+1)F^{(n)}(x)] \\ +m(x)[E_0(x)F(x)+E_1(x)F'(x)+ \cdots +E_n(x)F^{(n)}(x)] = 0. \end{aligned}$$

which may be written in the form

$$K_{n+1}(x)F^{(n+1)}(x) + K_n(x)F^{(n)}(x) + K_{n-1}(x)F^{(n-1)}(x) + \cdots = 0,$$

where

$$\begin{aligned} K_{n+1}(x) &= E_n(x + 1) + p(x)E_n(x), \\ K_n(x) &= E_{n-1}(x + 1) + p(x)E_{n-1}(x) + E_n'(x + 1) + p(x)E_n'(x) \\ &\qquad + q(x)E_n(x + 1) + m(x)E_n(x), \\ K_{n-1}(x) &= E_{n-2}(x + 1) + p(x)E_{n-2}(x) + E_{n-1}'(x + 1) + p(x)E_{n-1}'(x) \\ &\qquad + q(x)E_{n-1}(x + 1) + m(x)E_{n-1}(x + 1), \end{aligned}$$

and so forth. All of these K's must be zero since $f(x)$ satisfies the equation (1) for all values of $F(x)$. Hence

$$E_n(x + 1) = - p(x)E_n(x),$$

and

$$\begin{aligned} E_{n-1}(x + 1) = p(x)E_{n-1}(x) &= \frac{d}{dx}[p(x)E_n(x)] - p(x)E_n'(x) \\ &\qquad + p(x)q(x)E_n(x) - m(x)E_n(x) \\ &= p'(x)E_n(x) + p(x)q(x)E_n(x) - m(x)E_n(x) \\ &= - p(x)I(x)E_n(x). \end{aligned}$$

* Note that if $Js_n(x) = 0$ then $Is_{n-1}(x) = 0$, as may be seen by referring to § III.

Applying the transformation S to $f(x)$, we have

$$\begin{aligned} f_1^S(x) &= f(x+1) + p(x)f(x) \\ &= E_0(x+1)F(x) + E_1(x+1)F'(x) + \cdots + E_n(x+1)F^{(n)}(x) \\ &\qquad + p(x)[E_0(x)F(x) + E_1(x)F'(x) + \cdots + E_n(x)F^{(n)}(x)] \\ &= [E_n(x+1) + p(x)E_n(x)]F^{(n)}(x) \\ &\qquad + [E_{n-1}(x+1) + p(x)E_{n-1}(x)]F^{(n-1)}(x) + \cdots \\ &= 0 - p(x)I(x)F^{(n+1)}(x) + \cdots . \end{aligned}$$

We see that the order of the transformed expression in $F(x)$ is less than before. Repeat the process, reducing the order each time, until we get one of the invariants $I_k(x)$ zero, or else we get a new dependent variable

$$f_{S_\mu}(x) = R(x)F(x)$$

which satisfies the equation

$$f'_{S_\mu}(x+1) + p_{S_\mu}(x)f'_{S_\mu}(x) + q_{S_\mu}(x)f_{S_\mu}(x+1) + m_{S_\mu}(x)f_{S_\mu}(x) = 0,$$

or

$$\begin{aligned} R'(x+1)F(x) + R(x+1)F'(x) + p_{S_\mu}(x)[R'(x)F(x) + R(x)F'(x)] \\ + q_{S_\mu}(x)R(x+1)F(x) + m_{S_\mu}(x)R(x)F(x) = 0. \end{aligned}$$

This equation is an identity in $F(x)$, so the coefficients of $F(x)$ and of $F'(x)$ must be zero. Setting the coefficient of $F'(x)$ equal to zero, we get

$$p_{S_\mu}(x) = -\frac{R(x+1)}{R(x)}.$$

Putting this into the coefficient of $F(x)$ set equal to zero, we have

$$R'(x+1) - \frac{R(x+1)R'(x)}{R(x)} + q_{S_\mu}(x)R(x+1) + m_{S_\mu}(x)R(x) = 0.$$

Forming the invariant $I_{S_\mu}(x)$, we have

$$\begin{aligned} p(x)I_{S_\mu}(x) &= m_{S_\mu}(x) - p_{S_\mu}(x)q(x) - p'_{S_\mu}(x) \\ &= \frac{R(x+1)R'(x)}{R^2(x)} - \frac{R'(x+1)}{R(x)} - q_{S_\mu}(x)\frac{R(x+1)}{R(x)} \\ &\qquad + q_{S_\mu}(x)\frac{R(x+1)}{R(x)} + \frac{R(x)R'(x+1) - R(x+1)R'(x)}{R(x)} \end{aligned}$$

which is identically zero. Hence we see that $I_{S_\mu}(x) = 0$ is a necessary as well as a sufficient condition for a solution of the form

$$(14) \qquad f(x) = E_0(x)F(x) + E_1(x)F'(x) + \cdots + E_n(x)F^{(n)}(x).$$

Suppose that $J_{T_n}(x) = 0$* and $J_{T_k}(x) \neq 0$, where k takes the values $0, 1, 2, 3, \cdots, n-1$. The equation in $f_{T_n}(x)$ can then be written

* Note that if $I_{T_n}(x) = 0$, then $J_{T_{n-1}}(x) = 0$, as may be seen by referring to § III.

$$f'_{T_n}(x+1) + q_{T_n}(x) f_{T_n}(x+1) - [-p(x)][f'_{T_n}(x) + q_{T_n}(x) f_{T_n}(x)] = 0,$$

which, as was shown in § II, has a solution of the form

$$f_{T_n}(x) = e^{-\int_{x_0}^x q_{T_n}(x-1)dx} \int_{x_0}^x \theta(x) e^{\Sigma \log[-p(x)] + \int_{x_0}^x q_{T_n}(x-1)dx} + K e^{-\int_{x_0}^x q_{T_n}(x-1)dx},$$

where $\theta(x)$ is an arbitrary function of period one, and K is an arbitrary constant. We will write for the sake of brief notation

$$e^{-\Sigma \log[-p(x)]} f_{T_n}(x) = e^{\lambda(x)} V(x).$$

We have already developed the formula

$$(13) \qquad e^{-\Sigma \log[-p(x)]} f(x) = C_0 \Delta \{ C_1 \Delta [\cdots \Delta(C_n \Delta D_n)]\}.$$

In our present case

$$D_n = e^{\lambda(x)} V(x) \qquad \text{and} \qquad C_n = \frac{-1}{p(x) J_{T_{n-1}}(x)}.$$

Then $f(x)$ takes the form

$$(15) \quad f(x) = W_0(x)V(x) + W_1(x)V(x+1) + \cdots + W_n(x)V(x+n),$$

where the W's are determinate functions. Choosing $\theta(x) = 0$, we have the simpler integral

$$f(x) = K[W_0(x) + W_1(x) \cdots + W_n(x)].$$

or

$$f(x) = KW(x),$$

where $W(x)$ is a determinate function and K is an arbitrary constant.

The solution (15) we will call of rank $n+1$ of the *second kind*, and the equation for which $J_{T_n}(x) = 0$, and $J_{T_k}(x) \neq 0$, where k takes the values $0, 1, 2, 3, \cdots, n-1$, we will also call of rank $n+1$ of the second kind. In this paper, we will be concerned with the rank of the equation rather than with the rank of the solution.

VI. Equations of Finite Rank of the First Kind.

Suppose the equation (1) is of rank $n+1$ of the first kind. We then have

$$\alpha_{S_n}(x) = m_{S_n}(x) - p_{S_n}(x) q_{S_n}(x) - p'_{S_n}(x) = 0.$$

If $p_{S_n}(x)$ and $q_{S_n}(x)$ are chosen arbitrarily, $m_{S_n}(x)$ is defined by this equation. Using the expression for $m_{S_n}(x)$ thus defined, the invariant

$$J_{S_n}(x) = \frac{m_{S_n}(x)}{p_{S_n}(x)} - q_{S_n}(x)$$

becomes

$$J_{S_n}(x) = \Delta q_{S_n}(x) + \frac{d}{dx} \log p_{S_n}(x).$$

We have proved that

$$I_{S_{m+1}}(x) = I_{S_m}(x+1) + I_{S_m}(x) - J_{S_m}(x) - \Delta \frac{d}{dx} \log I_{S_m}(x),$$

and

$$J_{S_{m+1}}(x) = I_{S_m}(x),$$

from which we get

$$J_{S_{m-1}}(x) = J_{S_m}(x+1) + J_{S_m}(x) - I_{S_m}(x) - \Delta \frac{d}{dx} \log J_{S_m}(x).$$

Therefore, since $I_{S_n}(x) = 0$,

$$J_{S_{n-1}}(x) = J_{S_n}(x+1) + J_{S_n}(x) - \Delta \frac{d}{dx} \log J_{S_n}(x).$$

So we can now calculate the coefficients in backward succession.

Since $q_{S_{m+1}}(x) = q_{S_m}(x-1)$, we have

$$q(x) = q_{S_n}(x-n).$$

Also we have

$$p_{S_{n-1}}(x+1) = p_{S_n}(x) \frac{I_{S_{n-1}}(x)}{I_{S_{n-1}}(x+1)},$$

or

$$p_{S_{n-1}}(x+1) = p_{S_n}(x) \frac{J_{S_n}(x)}{J_{S_n}(x+1)},$$

also

$$\begin{aligned} p_{S_{n-2}}(x+2) &= p_{S_{n-1}}(x+1) \frac{J_{S_{n-1}}(x+1)}{J_{S_{n-1}}(x+2)} \\ &= p_{S_n}(x) \frac{J_{S_n}(x) J_{S_{n-1}}(x+1)}{J_{S_n}(x+1) J_{S_{n-1}}(x+2)}, \end{aligned}$$

. . . .

. . . .

$$p(x+n) = p_{S_n}(x) \frac{J_{S_n}(x) J_{S_{n-1}}(x+1) \cdots J_{S_1}(x+n-1)}{J_{S_n}(x+1) J_{S_{n-1}}(x+2) \cdots J_{S_1}(x+n)}.$$

Reducing the argument by n, we have

$$p(x) = p_{S_n}(x-n) \frac{J_{S_n}(x-n) J_{S_{n-1}}(x-n+1) \cdots J_{S_1}(x-1)}{J_{S_n}(x-n+1) J_{S_{n-1}}(x-n+2) \cdots J_{S_1}(x)}.$$

Also we have

$$\begin{aligned} m(x) &= p(x)q(x) + p'(x) + p(x)I(x) \\ &= p(x)q(x) + p'(x) + p(x)J_{S_1}(x), \end{aligned}$$

or we may use the form

$$m(x) = p(x)q(x-1) + p(x)J(x).$$

Thus we have developed the restrictions upon the coefficients of (1) which must exist if (1) is of finite rank of the first kind. That these conditions are also sufficient, may be seen by reversing the steps of the discussion.

VII. EQUATIONS OF FINITE RANK OF THE SECOND KIND.

Suppose the equation (1) is of rank $n+1$ of the second kind. Then we have

$$\beta_{T_n}(x) = m_{T_n}(x) - p_{T_n}(x)q_{T_n}(x-1) = 0.$$

Hence if $p_{T_n}(x)$ and $q_{T_n}(x)$ are arbitrarily given, $m_{T_n}(x)$ is determined. We have already found

$$J_{T_n}(x) = J_{T_{n-1}}(x) + J_{T_{n-1}}(x-1) - I_{T_{n-1}}(x-1) - \Delta \frac{d}{dx}\log I_{T_{n-1}}(x-1),$$

and

$$J_{T_{n-1}}(x) = I_{T_n}(x).$$

From these we get, remembering that $J_{T_n}(x) = 0$,

$$I_{T_n}(x) = J_{T_{n-1}}(x) = \frac{m_{T_n}(x)}{p_{T_n}(x)} - q_{T_n}(x) - \frac{p'_{T_n}(x)}{p_{T_n}(x)}$$
$$= -\Delta q_{T_n}(x-1) - \frac{p'_{T_n}(x)}{p_{T_n}(x)}.$$

Now we can calculate the coefficients in backward succession. We have seen that

$$p(x) = p_{T_n}(x).$$

We also have

$$q_{T_n}(x) = q_{T_{n-1}}(x-1) - \frac{d}{dx}\log\left[J_{T_{n-1}}(x)p(x)\right],$$

whence

$$q_{T_{n-1}}(x-1) = q_{T_n}(x) + \frac{d}{dx}\log\left[J_{T_{n-1}}(x)p(x)\right].$$

Hence it follows that

$$q_{T_n}(x-2) = q_{T_n}(x) + \frac{d}{dx}\log\left[J_{T_{n-1}}(x)J_{T_{n-2}}(x-1)p(x)p(x-1)\right],$$
$$q(x-3) = \,.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.$$
$$.\;.$$
$$q(x-n) = q_{T_n}(x) + \frac{d}{dx}\log\left[J_{T_{n-1}}(x)\cdots J(x-n)p(x)\cdots p(x-n)\right].$$

Increasing the argument by n, we have

$$q(x) = q_{T_n}(x+n) + \frac{d}{dx}\log\left[J_{T_{n-1}}(x+n)\cdots J(x)p(x+n)\cdots p(x)\right],$$

or

$$q(x) = q_{T_n}(x+n) + \frac{d}{dx}\log\left[I_{T_n}(x+n)\cdots I_{T_1}(x)p(x+n)\cdots p(x)\right].$$

To determine $m(x)$, we have

$$m(x) = p(x)q(x-1) + p(x)J(x),$$

or

$$m(x) = p(x)q(x-1) + p(x)I_{T_1}(x).$$

Thus we have found the restrictions on the coefficients of (1) which must hold if (1) is of finite rank of the second kind. By reversing the steps of the discussion we may see that these conditions are also sufficient. So we now have the necessary and sufficient conditions that (1) shall be of finite rank of either the first or the second kind.

VIII. Equations of Doubly Finite Rank.

The equation (1) is said to be of doubly finite rank when it is of finite rank with respect to both S and T. Suppose it is of rank $k + 1$ with respect to T, and of finite rank with respect to S. Transforming it k times by T, we have, since $J_{T_k}(x) = 0$,

$$(16)\quad f'_{T_k}(x + 1) + p(x)f'_{T_k}(x) + q_{T_k}(x)f_{T_k}(x + 1) + p(x)q_{T_k}(x - 1)f_{T_k}(x) = 0.$$

This equation is also of finite rank with respect to S. Suppose that rank is $r + 1$. We wish to see what restrictions are then imposed upon the coefficients $p(x)$ and $q_{T_k}(x)$.

First, apply the transformation

$$f_{T_k}(x) = v(x)h(x),$$

which does not change the rank. We will choose $v(x)$ so that the coefficient of $h(x + 1)$ in the new equation shall be zero. This requires that

$$\frac{v'(x + 1)}{v(x + 1)} = - q_{T_k}(x),$$

whence

$$v(x) = e^{-\int_{x_0}^{x} q_{T_k}(x-1)dx}.$$

The equation then becomes

$$h'(x + 1) + p(x)\frac{v(x)}{v(x + 1)}h'(x) = 0,$$

which may be written in the form

$$(17)\qquad h'(x + 1) + p(x)e^{\int_x^{x+1} q(x-1)dx}h'(x) = 0.$$

This equation, being of rank $r + 1$ with respect to S, has a solution of the form

$$h(x) = E_0(x)F(x) + E_1(x)F'(x) + \cdots + E_r(x)F^{(r)}(x),$$

where $F(x)$ is an arbitrary function of period one. Then $h'(x)$ has the form

$$h'(x) = Z_0(x)F(x) + Z_1(x)F'(x) + \cdots + Z_{r+1}(x)F^{(r+1)}(x),$$

where

$$Z_0(x) = E'_0(x), \qquad Z_1(x) = E_0(x) + E'_1(x), \qquad \cdots.$$

Substitute this value of $h'(x)$ in (17). Since it must satisfy (17) identically, we have the relations

$$Z_i(x+1) - R(x)Z_i(x) = 0, \tag{18}$$

where $R(x)$ is the negative of the coefficient of $h'(x)$ in (17).

Let $\overline{Z}(x)$ be any particular solution of (18). The other solutions may then be expressed in the form

$$Z_i(x) = w_i(x)\overline{Z}(x),$$

where the functions $w_i(x)$ have the period one. In particular

$$\begin{aligned}
Z_0(x) &= w_0(x)\overline{Z}(x) = E_0'(x),\\
Z_1(x) &= w_1(x)\overline{Z}(x) = E_0(x) + E_1'(x),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
Z_r(x) &= w_r(x)\overline{Z}(x) = E_{r-1}(x) + E_r'(x),\\
Z_{r+1}(x) &= w_{r+1}(x)\overline{Z}(x) = E_r(x).
\end{aligned}$$

From these we get

$$\begin{aligned}
E_r(x) &= w_{r+1}(x)\overline{Z}(x),\\
E_{r-1}(x) &= w_r(x)\overline{Z}(x) - \frac{d}{dx}[w_{r+1}(x)\overline{Z}(x)],\\
E_{r-2}(x) &= w_{r-1}(x)\overline{Z}(x) - \frac{d}{dx}[w_r(x)\overline{Z}(x)] + \frac{d^2}{dx^2}[w_{r+1}(x)Z(x)],\\
E_{r-3}(x) &= w_{r-2}(x)\overline{Z}(x) - \frac{d}{dx}[w_{r-1}(x)\overline{Z}(x)] + \frac{d^2}{dx^2}[w_r(x)\overline{Z}(x)]\\
&\qquad - \frac{d^3}{dx^3}[w_{r+1}(x)\overline{Z}(x)]\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
E_0(x) &= w_1(x)\overline{Z}(x) - \frac{d}{dx}[w_2(x)\overline{Z}(x)] + \cdots\\
&\qquad + \cdots + (-1)^r\frac{d^r}{dx^r}[w_{r+1}(x)\overline{Z}(x)].
\end{aligned}$$

Since $w_0(x)\overline{Z}(x) = E_0'(x)$, we have

$$w_0(x)\overline{Z}(x) = \frac{d}{dx}[w_1(x)\overline{Z}(x)] - \cdots + (-1)^r\frac{d^{r+1}}{dx^{r+1}}[w_{r+1}(x)\overline{Z}(x)],$$

which may be written in the form

$$\overline{Z}^{(r+1)}(x) + \eta_1(x)\overline{Z}^{(r)}(x) + \cdots + \eta_{r+1}(x)\overline{Z}(x) = 0. \tag{19}$$

The steps in the foregoing discussion are reversible. Hence it follows that the necessary and sufficient condition that the equation (1) shall be

of doubly finite rank is that

$$p(x)e^{\int_x^{x+1} q_{T_k}(x-1)dx} = \frac{\overline{Z}(x+1)}{\overline{Z}(x)},$$

where $\overline{Z}(x)$ is a solution of (19), in which the coefficients $\eta_i(x)$ involve the arbitrary periodic functions $w_i(x)$ as shown above.

IX. THE ANALOGUES OF LÉVY'S TRANSFORMATIONS.

In this section we consider generalizations of the Laplace-Poisson transformations, and investigate their usefulness in obtaining equations with vanishing invariants. These transformations are analogous to those applied by Lévy* to the analogous partial differential equation. They are

$$G(x) = f(x+1) + [p(x) + \gamma(x)]f(x)$$

and

$$H(x) = f'(x) + [q(x-1) + \delta(x)]f(x).$$

As the results are of a negative character, we shall merely state them without proof.

In case of the first transformation, the transformed equation cannot have a vanishing J-invariant unless we have $J(x) = 0$ from the original equation. The transformed equation cannot have a vanishing I-invariant except in very special cases.

In case of the second transformation, the transformed equation cannot have an I-invariant equal to zero unless the original equation gives $I(x) = 0$, and the J-invariant of the transformed equation cannot be zero except in very special cases.

So we have the result that the two transformations investigated in this section are not generally useful in obtaining an equation with a vanishing invariant.

UNIVERSITY OF ILLINOIS,
May, 1918.

* *Journal de l'École Polytechnique*, t. 38 (1886), p. 67.

CHARACTERISTIC SUBGROUPS OF AN ABELIAN PRIME POWER GROUP.

BY G. A. MILLER.

§ 1. *Introduction.*

A subgroup which corresponds to itself in every possible automorphism of a given group is called a characteristic subgroup or an I-invariant subgroup. Some fundamental properties of the characteristic subgroups of any abelian group were studied by the writer of the present article in a paper published in volume 27 of the AMERICAN JOURNAL OF MATHEMATICS, 1905, pages 15–24. In particular, it was noted in this paper that besides the identity there is a certain characteristic subgroup, called the fundamental characteristic subgroup, which appears in every possible characteristic subgroup of an abelian group G whose order is of the form p^m, p being some prime number.

The present paper is devoted to a determination of various new properties of the characteristic subgroups of G. For the sake of clearness it seems desirable to explain here a few terms which are frequently employed. Two groups H_1 and H_2 are said to be complementary groups as regards a group G provided at least one H_1 of these two groups is an invariant subgroup of G while the other H_2 is simply isomorphic with the quotient group of G with respect to H_1. When G is abelian it is known that H_2 is simply isomorphic with at least one subgroup of G and hence it is convenient to speak of *complementary subgroups of G. Two invariant subgroups H_1 and H_2 are said to be complementary subgroups of a group G provided each of these subgroups is simply isomorphic with the quotient group of G with respect to the other.*

It should first be noted that if one of two subgroups of G is simply isomorphic with the quotient group of G with respect to the other the two subgroups are not necessarily complementary. For instance, every subgroup of order p is simply isomorphic with each of the quotient groups arising from the subgroups of index p, but when G involves λ distinct invariants the quotient groups arising from its subgroups of order p are of λ distinct types. Each of these types corresponds to one set of I-conjugate subgroups of order p. That is, there are just λ sets of I-conjugate complementary subgroups of order and of index p. It will be seen in the following section that these correspond to the λ characteristic subgroups generated by operators of order p, and the λ characteristic subgroups involving oper-

ators of order p^{α_1-1}, p^{α_1} being the order of the largest operators contained in G.

The complementary subgroup of the fundamental characteristic subgroup is composed of all the operators of G whose orders divide p^{α_1-1} and it is characterized by the fact that it is the only characteristic subgroup of G which includes every other characteristic subgroup of G. It is the cross-cut of all the subgroups of index p under G which are complementary to the subgroups of order p contained in the fundamental characteristic subgroup of G. It should be noted that the numbers of these complementary subgroups of order and of index p are equal to each other.

The simplest characteristic subgroups of G are those composed of all the operators of G whose orders divide p^{β}, $\beta < \alpha_1$. The complementary subgroup of such a characteristic subgroup is composed of the p^{β} power of every operator of G. A necessary and sufficient condition that there are no other characteristic subgroups in G is that all the invariants of G are equal to each other. In this case, the number of the characteristic subgroups, besides the identity, is therefore equal to $\alpha_1 - 1$, and the number of pairs of complementary characteristic subgroups is $(\alpha_1 - 1)/2$ when α_1 is odd. When α_1 is even, the number of these pairs is $(\alpha_1 - 2)/2$ and one of the characteristic subgroups is self-complementary.

It may be desirable to direct attention to the difference between complementary subgroups and subgroups of complementary types. If G is of type $(m_1, m_2, \cdots, m_\lambda)$, and if two of its subgroups are of types $(\alpha_1, \alpha_2, \cdots, \alpha_{\lambda_1})$ and $(\beta_1, \beta_2, \cdots, \beta_{\lambda_2})$ respectively, these subgroups are said to be of complementary types when it is possible to satisfy each of the following equations, where x is either 0 or some α, and y is either 0 or some β and each α or β is used only once*:

$$x + y = m_i \qquad (i = 1, 2, \cdots, \lambda).$$

Subgroups which are of complementary types are clearly also complementary subgroups. That the converse is not necessarily true may be seen by considering the group of type (4, 1). This group contains operators of order p^3 which are not powers of operators of order p^4 and such an operator of order p^3 generates a subgroup whose complementary subgroups are cyclic and of order of p^2 but are not contained separately in cyclic groups of order p^4. Hence $\alpha_1 = \alpha_{\lambda_1} = 3$ and $\beta_1 = \beta_{\lambda_2} = 2$ in the present case, so that neither of the two equations $x + y = m_i$, $(i = 1, 2)$, can be satisfied. The complementary subgroup of the cyclic subgroup of order p^3 which is contained in the cyclic subgroups of order p^4 is of type (1, 1), and in this case the complementary subgroups are also of complementary types.

* G. A. Miller, *Transactions of the American Mathematical Society*, vol. 21 (1920), p. 313.

§ 2. *Characteristic subgroups generated by operators of a given order.*

The number of the characteristic subgroups contained in G depends on the number of the different invariants of G but is not affected by the number of these invariants which are equal to each other. That is, if no two of the invariants of G are equal to each other G has exactly the same number of characteristic subgroups as the group G which includes G and has no invariant except such as are equal to those of G, but which has at least two equal invariants. Hence in the study of the number of the characteristic subgroups of G it may be assumed, without loss of generality, that G is of type $(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$, $\alpha_1 > \alpha_2 > \cdots > \alpha_\lambda$. To emphasize the fact that some invariants may be equal to each other the group G will be replaced by the group G'. Let $\alpha_1 - \alpha_2 = \alpha'_1$.

The cyclic subgroup of order p^r, $r < \alpha'_1$, which is generated by each operator of order p^{α_1} contained in G is evidently a characteristic subgroup of G, and G contains no other cyclic characteristic subgroup whenever $p > 2$. When $p = 2$ and $1 < r < \alpha'_1$, G clearly has two and only two cyclic characteristic subgroups of order p^r. Hence the following theorem: *An abelian group of order p^m cannot have more than one characteristic subgroup of order p. A necessary and sufficient condition that such a group G contains at least one characteristic cyclic subgroup of order p^r, p^r being less than $p^{\alpha'_1}$, is that G' has only one largest invariant. A necessary and sufficient condition that G' contains two cyclic characteristic subgroups of order p^r, $1 < r < \alpha'_1$, is that $p = 2$ and that G' has only one largest and also only one next to the largest invariant. No abelian group of order p^m has more than two characteristic cylic subgroups of the same order.*

From the preceding theorem it results that there is a marked difference as regards characteristic subgroups between the groups whose orders are powers of 2 and those whose orders are powers of an odd prime number. Hence we shall assume in what follows, unless the contrary is stated, that $p > 2$. It is easy to prove that every characteristic subgroup of G which involves operators of order p^r must involve the subgroup generated by all the operators of order p^r which are contained in the cyclic subgroups of order p^{α_1} found in G. Hence this characteristic subgroup will be called *the fundamental characteristic subgroup generated by operators of order p^r.* The fundamental characteristic subgroup noted in the first paragraph of the Introduction may therefore also be called, in accord with this more general nomenclature, the fundamental characteristic subgroup generated by operators of order p.

A characteristic subgroup of G cannot involve any operator of order p^{α_1} since all the operators of highest order contained in G are I-conjugate and every abelian group is generated by its operators of highest order. Every

characteristic subgroup of G which involves operators of order p^{α_1-1} must involve the ϕ-subgroup of G since this is composed of all the operators of G which have the property that each of them is the pth power of some other operator of G. Hence the ϕ-subgroup of G is its fundamental characteristic subgroup involving operators of order p^{α_1-1}. When $\lambda > 1$, G has more than one characteristic subgroup which are generated by operators of order p^{α_1-1} These can be arranged linearly so that each contains all those which precede it and involves one more set of I-conjugate operators of order p^{α_1-1} than the one which immediately precedes it.

In fact, the first of these characteristic subgroups is the ϕ-subgroup of G and the remaining $\lambda - 1$ may be obtained by adjoining successively all the operators of smallest order contained in G which are not found in the preceding characteristic subgroup. The complementary subgroups of these characteristic subgroups taken in the reverse order are the λ characteristic subgroups of G which are generated by its operators of order p, and the sum of the numbers of the sets of I-conjugate operators of highest order in each pair formed by one of these characteristic subgroups and its complementary subgroup is $\lambda + 1$. These complementary subgroups are evidently also of complementary types.

All the subgroups of index p under G have the ϕ-subgroup of G for their cross-cut. Hence each such subgroup corresponds to a subgroup of index p in the ϕ-quotient group of G. These subgroups may be divided into λ sets of I-conjugate subgroups corresponding to the λ characteristic subgroups of G which involve only operators of order p. Hence the following theorem: *In any group of order p^m, p being a prime number, all the subgroups of index p which are of the same type are I-conjugate.* It may be noted that it is possible to construct prime power abelian groups in which there are subgroups of every other index which are not I-conjugate. In fact, the abelian group of order p^m and of type $(m - 1, 1)$, $m > 2$, contains cyclic subgroups of every index $> p$ which are not I-conjugate.

A direct proof of the italicized theorem of the preceding paragraph is as follows: The independent generators of G' can be so selected that they differ from the possible independent generators of any given subgroup of index p only as regards one operator, and that the pth of this operator is the remaining independent generator of the subgroup in question. If such a selection of the independent generators of two subgroups of index p and of the same type is made a (1, 1) isomorphism between these subgroups may be established so as to make an independent generator of one of these subgroups correspond to an arbitrary independent generator of the same order in the other. Hence these subgroups correspond in some automorphism of G.*

* Miller, Blichfeldt, Dickson, *Finite Groups*, 1916, p. 73.

It was noted above that the λ characteristic subgroups of G which are generated by operators of order p^{α_1-1} can be arranged linearly so that each includes all those which precede it and that no two of these characteristic subgroups are of the same type. It will be seen that *no abelian group of order p^m, p being an odd prime number, contains two characteristic subgroups which are of the same type,* but it is not always possible to arrange linearly the characteristic subgroups which are generated by operators of order p^r so that each of these subgroups includes all those which precede it.

To obtain all the characteristic subgroups of G which are generated by operators of order p^r we may begin with the fundamental characteristic subgroup K_r of G which is generated by operators of order p^r. In this characteristic subgroup all the operators of order p^r constitute a single set of I-conjugates. When $\lambda > 1$, a characteristic subgroup whose largest operators are of order p^r and which involves two sets of I-conjugate operators of this order can be obtained by adjoining to K_r the smallest set of I-conjugate operators of lowest order found in G but not in K_r. When this lowest order is p there may be more than one set of I-conjugate operators of lowest order found in G which are not contained in K_r. In this case such sets are added successively in order of magnitude beginning with the smallest. We thus obtain a series of characteristic subgroups which can be arranged linearly so that each includes all those which precede it.

A new series of characteristic subgroup may be started by adjoining to K_r the smallest set of I-conjugate operators of lowest order found in G but not contained in the last subgroup of the preceding series. The first subgroup of this new series K'_r involves two or three sets of I-conjugate operators of order p^r according as it does not or does contain more operators of lowest order than K_r. To obtain the various characteristic subgroups of the second series we adjoin to K'_r the smallest set of I-conjugate operators of lowest order found in G but not in K_r. As all the characteristic subgroups of G can be found by continuing this process it has been proved that no two characteristic subgroups of G are of the same type.

It results from this method for finding all the characteristic subgroups of G that while the number of the characteristic subgroups of G cannot exceed the number of the different types of subgroups found in G it may be less than this number. A necessary and sufficient condition that G contains a characteristic subgroup of type $(r_1, r_2, \cdots, r_\lambda)$, where $r_1 \geqq r_2 \geqq \cdots \geqq r_\lambda$ and one or more of the r's may be 0 is that $r_\gamma - r_{\gamma+1} < \alpha_\gamma - \alpha_{\gamma+1}$ $(\gamma = 1, 2, \cdots, \lambda - 1)$ whenever r_γ and $r_{\gamma+1}$ are different from 0. Hence the following theorems: *The number of the characteristic subgroups of any abelian group G' of order p^m, p being an odd prime number, is equal to the number of the characteristic subgroups in a subgroup G of G' which has for*

its independent generators all the different independent generators of G' but has no two independent generators of the same order. If G is of type $(\alpha_1, \alpha_2, \cdots, \alpha_\lambda)$ then G contains one and only one characteristic subgroup of type $(r_1, r_2, \cdots, r_\lambda)$, $r_1 \geqq r_2 \geqq \cdots \geqq r_\lambda$, where $r_1 < \alpha_1$ and some of the r's may be 0, and $r_\gamma - r_{\gamma+1} \leqq \alpha_\gamma - \alpha_{\gamma+1}$ $(\gamma = 1, 2, \cdots, \lambda - 1)$ whenever r_γ and $r_{\gamma+1}$ are both different from 0.

To illustrate this theorem it may be noted that the abelian group of order p^{10} and of type 3, 3, 2, 1, 1 contains one and only one characteristic subgroup of each of the following types: (1, 1), (1, 1, 1), (1, 1, 1, 1, 1), (2, 2, 1), (2, 2, 1, 1, 1), (2, 2, 2, 1, 1). Hence this group contains six characteristic subgroups besides the identity, and this is also the number of the characteristic subgroups of the abelian group of order p^6 and of type 3, 2, 1 whenever $p > 2$. These characteristic subgroups are of the following types: (1), (1, 1), (1, 1, 1), (2, 1), (2, 1, 1), (2, 2, 1).

It should be noted that every characteristic subgroup of G' has a complementary characteristic subgroup and this is also of the complementary type. When the characteristic subgroups of G' whose largest operator is of order p^r can be arranged linearly so that each includes all those which precede it their complementary characteristic subgroups can be arranged linearly so that each is included in all those which follow it. In particular, the largest characteristic subgroup whose largest operators are of order p^r has for its complementary subgroup the smallest characteristic subgroup whose largest operators are of order $p^{\alpha_1 - r}$, and vice versa.

In view of the reciprocal relations between the characteristic subgroups of G' it may be assumed without loss of generality that $2r \leqq \alpha_1$. From what precedes there results the following theorem: *The number of the different sets of I-conjugate operators of highest order in any characteristic subgroup whose largest operators are of order p^r, increased by the number of the different sets of I-conjugate operators of highest order in its complementary characteristic subgroup, is equal to one more than the number the characteristic subgroups whose largest operator is of order p^r.* In particular, the sum of the numbers of the different sets of *I*-conjugate operators of highest order in any characteristic subgroup and its complementary characteristic subgroup is independent of the choice of the former characteristic subgroup when the order of the operators of highest order is fixed.

Number of characteristic subgroups when G is of type (1, 2, 3, $\cdots$, m).

In order to exhibit clearly a method for determining all the characteristic subgroups of any abelian group it seems desirable to consider separately the special case when G is of type (1, 2, 3, $\cdots$, m) since the formula representing the total number of these subgroups is comparatively simple in this

case. It will be convenient to assume that m may represent an indefinitely large number and to consider separately all the characteristic subgroups, whose operators of highest order is p^r. When r is 1 it is evident that there are m such subgroups and that the orders of these r subgroups are p, p^2, $\cdots$, p^m. The number of sets of I-conjugate operators contained in each of these subgroups is 1, 2, $\cdots$, m respectively. That is, this number is equal to the index of p representing the order of the characteristic subgroup.

When $n = 2$ the smallest characteristic subgroup is of order p^3, since this is the order of the fundamental characteristic subgroup K_2 generated by operators of order p^2. All the operators of order p^2 contained in K_2 are I-conjugate since they are separately powers of operators of order p^m contained in G. This characteristic subgroup involves two of the characteristic subgroups whose operators of highest order are of order p, and if we adjoint to K_2 successively the latter characteristic subgroups which are of orders p^3, p^4, $\cdots$, p^m respectively there result $m - 1$ characteristic subgroups whose operators of highest order are of order p^2. Each of these $m - 1$ characteristic subgroups has only one independent generator of highest order, and each of these subgroups involves one more complete set of I-conjugate operators of order p^2 than the one which precedes it.

The smallest characteristic subgroup of G which has two independent generators of highest order and involves no operator whose order exceeds p^2 is of order p^5, and involves three of the characteristic subgroups of G' which are separately generated by operators of order p. It involves three sets of I-conjugate operators of order p^2. This characteristic subgroup can be extended by means of characteristic subgroups generated by operators of order p just as K_2 was extended except that the first of these extending subgroups is of order p^4. Each such extension increases by two the number of I-conjugate operators of order p^2, and hence the number of the sets of I-conjugate operators of order p^2 found in the last one of these characteristic subgroups is equal to the total number of the characteristic subgroups continued in G and having no more than two independent of highest order, viz., p^2. This number is $m - 1 + m - 2$.

As this process may be continued until all the characteristic subgroups generated by operators of order p^2 have been found it results that the number of such characteristic subgroups which involve exactly α independent generators of order p^2 is $m - \alpha$, $\alpha = 1, 2, \cdots, m - 1$. The total number of these characteristic subgroups is therefore

$$m(m - 1)/2.$$

The total numbers of the characteristic subgroups generated by the operators of orders p and p^2 contained in G are therefore the sums of the terms,

of the following series of figurate numbers of orders 0 and 1 respectively:

$$1, 1, 1, \cdots \text{ to } m \text{ terms,}$$

$$1, 2, 3, \cdots \text{ to } m - 1 \text{ terms.}$$

The fundamental characteristic subgroup K_3 of G generated by operators of order p^3 is of order p^6, and involves three characteristic subgroups generated by operators of order p as well as three such subgroups generated by operators of order p^2. The number of the characteristic subgroups which involve K_3 but have only two independent generators of order p^2 is evidently $m - 2$ and these subgroups involve $1, 2, \cdots, m - 2$ sets of I-conjugate operators of order p^3 respectively. The number of the characteristic subgroups of G which involve only one independent generator of order p^3 but three and only three independent generators of order p^2 is $m - 3$, etc. Hence the number of the characteristic subgroups of G which involve K_3 but have separately only one independent generator of order p^3 is the sum of the series $1, 2, \cdots, m - 2$. Similarly it results that the number of the characteristic subgroups of G which involve K_3 and have separately two and only two independent generators of order p^3 is the sum of the series $1, 2, \cdots, m - 3$.

As this process may be continued until the largest characteristic subgroup of G which is generated by operators of order p^3 has been reached, it results that the number of the characteristic subgroups of G which can be generated by operators of order p^3 is the sum of the figurate numbers of the second order, terminating with $m - 2$. The three special cases considered thus far suggest the following theorem: *The number of the characteristic subgroups of G which are separately generated by operators of order p^r is the sum of the figurate numbers of order $r - 1$, terminating with $m - r + 1$.*

This theorem can easily be proved by mathematical induction since the fundamental characteristic subgroup K_r of G generated by operators of order p^r involves r of the characteristic subgroups generated by operators of order p. Those characteristic subgroups of G which involve only one independent generator of order p^r can be found in the same manner as the characteristic subgroups generated by operators of order p^{r-1} were found with the exception that the series of numbers in the present case consists of $m - r + 1$ numbers while in the preceding case it consisted of $m - r + 2$. The sum of the first $m - r + 1$ figurate numbers of order $r - 2$ which constitute the preceding series is therefore the last term of this series.

Similarly, the sum of the first $m - r$ figurate numbers of order $r - 2$ is the next to the last term in the present series, etc. Hence it results that the present series is composed of figurate numbers if the preceding series was

composed of such numbers, and hence the proof of the theorem in question, is complete. The fact that the number of the characteristic subgroups generated by operators of order p^r is equal to the number of such subgroups generated by operators of order p^{m-r} follows directly from the properties of figurate numbers* as well as from the (1, 1) correspondence between the characteristic subgroups of complementary types.

In the special case under consideration it is easy to see that the number of the characteristic subgroups generated by operators of order p^r is equal to the number of the sets of I-conjugate operators of this order. Hence the latter number is also the sum of the figurate numbers of order $r-1$ when G is of type $(1, 2, 3, \cdots, m)$. As was noted above this result is not affected when G has more than one invariant which is equal to p^α, $1 \leqq \alpha \leqq m$.

* Cf., P. Bachmann, *Niedere Zahlentheorie*, 1910, p. 10.

ERRATA.

Page 152, in the determinants D_1, D_2, and D_3, in place of $h-1$, $i-1$, $2i-h-1$, etc., read a_{h-1}, a_{i-1}, a_{2i-h-1}, etc.

Page 153, the words "the following figures" immediately above the figures refer to the three lower figures. The two upper figures should be three lines higher up, at the beginning of the paragraph; and the inscriptions "Vanishing parallelogram for n odd," and "Vanishing parallelogram for n even," belong to these two upper figures respectively.

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